Generating Functions For $P^0(N)$ And $P^d(N)$

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Abstract

In this article number of partitions of n into odd parts and the number of partitions of n into distinct parts are discussed.

1. Introduction

In this paper we have discussed the generating functions for $p^0(n)$ and $p^d(n)$. These functions are collected from S. Ramanujan's lost Notebook (Hardy and Wright 1965). We have analyzed theses functions with algebraic calculations, theorem and numerical example. We have proved that both functions are equal.

2. Generating Function for $p^0(n)$

We consider the generating function for $p^o(n)$, which is given by (Hardy and Wright 1965, Burn 1964);

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots\infty}$$

$$= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \ldots \infty$$

$$= 1 + \sum_{n=1}^{\infty} p^o(n)x^n, \qquad (1)$$

Where the coefficient $p^o(n)$ is the number of partitions of n into odd parts.

3. Generating Function for $p^d(n)$

The generating function for $p^d(n)$ is of the form (Hardy and Wright 1965, Burn 1964);

$$(1 + x)(1 + x^2)(1 + x^3)\ldots\infty$$

$$= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \ldots\infty$$

$$= 1 + \sum_{n=1}^{\infty} p^d(n)x^n, \qquad (2)$$

Where the coefficient $p^d(n)$ is the number of partitions of *n* into distinct parts.

Now from (1) & (2) we have;

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$$1+\sum_{n=1}^{\infty} p^{o}(n)x^{n} = \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots\ldots\infty} = \frac{(1-x^2)(1-x^4)(1-x^6)}{(1-x)(1-x^2)(1-x^3)}\ldots.\infty$$

$$= (1+x)(1+x^2)(1+x^3)\ldots\infty = 1+\sum_{n=1}^{\infty} p^{d}(n)x^{n}.$$

Now equating the coefficient of x^n from both sides we get;

$$p^0(n)= p^d(n).$$

Theorem 1: $p^0(n)= p^d(n)$ i.e. the number of partitions of n into odd parts is equal to the number of partitions of n into unequal parts.

Proof: We develop an one to one correspondence between the partitions enumerated by $p^0(n)$ and those enumerated by $p^d(n)$. We start with any partition of n into odd parts say $n = a_1 + a_2 + \ldots.+a_r$. Among these *r* odd integers, suppose there are m distinct ones, say $c_1, c_2, \ldots\ldots c_m$ by rearranging notation if necessary. Collecting like terms in the partition of n, we get $n= e_1c_1+e_2c_2+\ldots\ldots+e_mc_m$. We write each co-efficient e_i as a unique sum of distinct powers of 2, and write each e_ic_i as a sum of terms of the type 2^tc_i. This gives n as a partition into distinct parts. Thus we have the one to one correspondence.

Such as a number k can be expressed uniquely in the binary scale

i.e. as $k = 2^a + 2^b + 2^c + \ldots$ ($o \le a < b < c \ldots$).

Hence a partition of n into odd parts can be written as $n = k_1.1 + k_2.3 + k_3.5+ \ldots$

$= (2^{a1} + 2^{b1} + ---).1 + (2^{a2} + 2^{b2}+ --).\ 3 + (2^{a3} + ---).\ 5 + ----$
and there is an one to one correspondence between this partitions and the into distinct parts $2^{a1}.1, 2^{b1}.1, \ldots\ldots., 2^{a2}.3, 2^{b2}.3, \ldots.., 2^{a3}.5, \ldots\ldots..$

Conversely let $n = a_1 + a_2 + \ldots.+a_r$ be a partition of n into distinct parts. We convert this partition into a partition of n with odd parts. For any even positive integer m can be expressed 2^j f(m) as a multiple odd integer. As for example $4 = 1.2^2$, $6 = 3.2^1$, $10 = 4 + 6 = 1.2^2 + 3.2^1$, where 2^j is the highest power of 2 and f(m) is an odd integer. Suppose there are distinct odd integers among $f(a_1), f(a_2),.., f(a_r)$.

Rearrange the subscripts of necessary so that $f(a_1), f(a_2),\ldots, f(a_s)$ are distinct odd integers, and $f(a_{s+1}), f(a_{s+2}),\ldots, f(a_r)$ are duplicates of these. Collecting terms, we can write $n = \sum_{i=1}^{s} c_i f(a_i)$ with positive integers coefficients c_i. The final step is to write each $c_i f(a_i)$ in the form $f(a_1) + f(a_2) +\ldots+f(a_i)$ with c_i terms in the sum. Thus n is expressed as a sum of odd integers. clearly our correspondence is onto so that $p^0(n)= p^d(n)$.

4. A Numerical Example

We take *n* = 11, the list of partitions of 11 into odd parts is given below:

11 = 9+1+1 = 7+3+1 = 7+1+1+1+1 = 5+5+1

=5+3+1+1+1=5+1+1+1+1+1+1=5+3+3=3+3+3+1+1

= 3+3+1+1+1+1+1 = 3+1+1+1+1+1+1+1+1

= 1+1+1+1+1+1+1+1+1+1+1.

So there are 12 partitions i.e. $P^o(11) = 12$.

Again the list of partitions of 11 into unequal parts is given below

11 =10+1 = 9+2 = 8+3 = 8+2+1 = 7+4 = 7+3+1

= 6+5 = 6+4+1 = 6+3+2 = 5+4+2= 5+3+2+1.

So there are 12 partitions i.e. $P^d (11) = 12$.

Therefore, $P^0(11) = P^d(11)$.

5. Conclusion

In this study we have found the partitions of n into odd parts and the partitions of n into distinct parts given by the coefficients of x^n from above functions respectively.

Reference

R.P. Burn, A Path Way into Number Theory, Cambridge University Press, Second Edition, 1964.

G.H, Hardy and E.M. Wright, Introduction to the Theory of Numbers, 4th Edition, Oxford, Clarendon Press, 1965.