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A New Fuzzy Regression Model by Mixing Fuzzy and Crisp Inputs

Magda M. M. Haggag[1]

Abstract

This paper proposes a new form of the multiple regression model (mixed model) based on adding both fuzzy and crisp input data. The least squares approach of the proposed multiple regression parameters are derived in different cases. This derivation is based on the fact that each fuzzy datum is a nonempty compact interval of the real line. The main contribution is to mix both fuzzy and crisp predictors in the linear regression model. The mixed fuzzy crisp model will be introduced mathematically and by coded via R-language. The least squares of the regression parameters will be derived and evaluated using distance measures. Numerical examples using generated data showed best results for the mixed fuzzy crisp multiple regression models compared to the multiple fuzzy models.

Keywords: Bertolouzza distance, Compact data sets, Euclidean distance, Fuzzy least squares, Fuzzy variables, Fuzzy regression, tight data**.**

(1) Introduction

Linear regression models are used to model the functional relationship between the response and the predictors linearly. This relationship is used for describing and estimating the response variable from predictor variables. Some important assumptions are needed to build a relationship, such as existing enough data, the validity of the linear assumption, the exactness of the relationship, and the existence of a crisp data for variables and coefficients.

The fuzzy regression model is a practical alternative if the linear regression model does not fulfill the above assumptions. A fuzzy linear regression model first introduced by Tanaka et al. (1982). Their approach handled after that by many authors, such as Tanaka and Lee (1988); Tanaka and Watada (1988); Tanaka et al. (1989); Diamond (1988, 1990, 1992); Diamond and Koener (1997); D'Urso and Gastaldi (2000); Yang and Lin (2002); D'Urso (2003); Gonzalez-Rodriguez et al. (2009); Choi and Yoon (2010); Yoon and Choi (2009, 2013); D'Urso and Massari (2013).

Fuzzy regression models have been treated from different points of view depending upon the type of input and output data. There are three different kinds of models:

- Crisp input and fuzzy output with fuzzy coefficients.
- Fuzzy input and fuzzy output with crisp coefficients.
- Fuzzy input and fuzzy output with fuzzy coefficients.

The least squares method is used to estimate the fuzzy regression model. (See for instance, Diamond (1988, 1990, 1992)).

The objective of this paper is to extend the simple linear regression model to the multiple one and estimate it with the least squares approach. This extension is based on adding both fuzzy and crisp predictors to the linear regression model, and the resulting model is called the mixed fuzzy crisp (MFC).

[1] Associate Professor of Statistics, Head of the Department of Statistics, Mathematics, and Insurance, Faculty of Commerce, Damanhour University, Egypt. Email: magda.haggag@com.dmu.edu.eg & magmhag@yahoo.com

Our extended model will be evaluated using the extended squared distance of Diamond (1988). Generated data are applied to compare the estimation results of the proposed MFC model with the usual multiple fuzzy MF regression model.

This paper will be outlined as follows. Section (2) presents some definition regarding fuzzy random variables (FRVs), fuzzy distance and possibility distributions will be introduced. In section (3) fuzzy linear regression models will be considered. The proposed mixed fuzzy and crisp (MFC) linear regression model will be introduced in section (4). Section (5) considers the numerical applications using generated and real data examples. The concluding remarks will be discussed in section (6).

(2) Mathematical Preliminaries

Some definitions and notes will be presented in this section for the requirements of this work.

2.1 Sets Representation of Fuzzy Numbers

Let $K_c(R^p)$denotes the class of all non-empty compact intervals of R^p and let $F_c(R^p)$ denotes the class of all fuzzy numbers of R^p. Then, $F_c(R^p)$ will be defined as follows:

$$F_c(R^p) = \{A : R^p \to [0,1] \mid A_\alpha \in K_c(R^p) \forall \alpha \in [0,1]\}, \tag{1}$$

where A_α is the α-cut set of A if $\alpha \in (0,1]$, and A_0 is called the support of A. (Zadeh, 1975).

For a given $A, B \in F_c(R)$, and $b \in R$, the followings hold:

- The sum of A and B is called the Minkowski sum, defined as: $S = A \oplus B \in F_c(R)$. (Zadeh, 1975).
- The scalar product of b and the set A is defined as: $P = b \otimes A \in F_c(R)$. (Zadeh, 1975).
- A fuzzy number $D \in F_c(R)$ is called the Hukuhara difference of A and B defined as: $D = A -_H B$, it is shown that the Hukuhara difference is the inverse operation of addition $\oplus$, where $A = B \oplus D$.(Zadeh, 1975).

2.2 Left and Right (L-R) Representation of Fuzzy Numbers

Let A∈T(R) is a FRV, where T(R) is a set of trapezoidal fuzzy numbers of $F_c(R)$. A trapezoidal fuzzy number A is defined as $A=Tra(A_l,A_u,A_v,A_r)$, where $A_l \in R$ *and* $A_r \in R$ *are* the left and right limits of the trapezoidal fuzzy number A, respectively. Also $A_u \in R$ *and* $A_v \in R$ *are* the left and right middle points of A, respectively, as shown in Figure (1). When $A_u = A_v = A_m$, a fuzzy number A will be a triangular, i.e., $A=Tri(A_l,A_m,A_r)$, as shown in Figure (2)

If $A_l=a$, $A_u=b$, $Av=c$, and $A_r=d$, a stylized representation of a trapezoidal fuzzy number A can be represented in the following L-R form:

- A trapezoidal fuzzy number A is specified by a shape function with the following membership (Figure (1)):

$$\mu_A(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x \le b \\ 1, & b \le x \le c \\ \dfrac{c-x}{d-c}, & c < x \le d \\ 0, & x \ge d \end{cases}$$

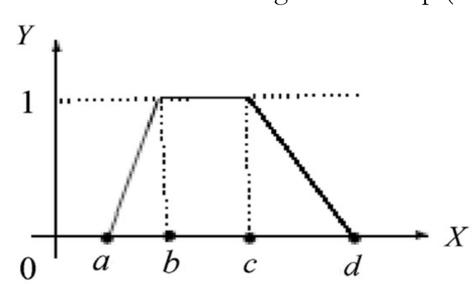

Figure (1): Trapezoidal Fuzzy Number.

- When c=b, a triangular fuzzy number A is specified by a shape function with the following membership (Figure (2)):

$$\mu_A(x)=\begin{cases}0, & x\le a\\ \dfrac{x-a}{b-a}, & a<x\le b\\ 1, & x=b\\ \dfrac{c-x}{c-b}, & b<x\le c\\ 0, & x\ge c.\end{cases}$$

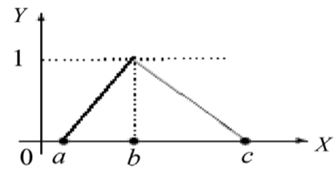

Figure (2): Triangular Fuzzy Number

2.3 Metrics in Fuzzy Numbers Space

To measure the distance between any two fuzzy numbers A, and B in $F_c(R)$, an extended version of the Euclidean (L_2) distance ($d_E(A,B)$) is defined by:

$$d_E^2(A,B)=\int_0^1[A_L(\alpha)-B_L(\alpha)]^2\,d\alpha+\int_0^1[A_U(\alpha)-B_U(\alpha)]^2\,d\alpha\,, \qquad (4)$$

where $A_L(\alpha)$ and $A_U(\alpha)$ are the lower and upper α -cuts of a fuzzy number A. (Grzegorzewski, 1998).

Bertoluzza et al. (1995) have proposed the so-called Bertoluzza metric d(A,B), which is defined as:

$$d^2(A,B)=\int_{[0,1]}[mid(A_\alpha)-mid(B_\alpha)]^2\,d\alpha+\int_{[0,1]}[spr(A_\alpha)-spr(B_\alpha)]^2\,d\alpha\,, \qquad (5)$$

where $mid(A_\alpha)=\dfrac{A_\alpha^U+A_\alpha^L}{2}$ denotes the midpoint of A_α, and $spr(A_\alpha)=\dfrac{A_\alpha^U-A_\alpha^L}{2}$ denotes the spread (or radius) of A_α, $\forall\alpha\in[0,1]$. A_α^U and A_α^L denote the upper bound and lower bound of A, respectively.

The Hausdroff $d_H(A,B)$ metric for $A,B\in F_c(R)$ is given by:

$$d_H(A,B)=\max\{|\inf A-\inf B|,|\sup A-\sup B|\}, \qquad (6)$$

where *infA* is the infimum value of A, and *supA* is the supremum value of A.

The $d_p(A,B)$ metric for $A,B\in F_c(R)$, and $1\le p\prec\infty$ is given by:

$$d_p(A,B)=\left\{\frac{1}{2}|\inf A-\inf B|^p+\frac{1}{2}|\sup A-\sup B|^p\right\}^{1/p}, \qquad (7).$$

where *infA* and *supA* are the infimum and supremum values of A, respectively. (See Vitale, 1985).

The distance between fuzzy numbers can be defined as the distance between their membership functions. The distance $d_p(A,B)$ between the two fuzzy numbers *A,B* is given by:

$$d_p(A,B)=\int_X\left[|\mu_A-\mu_B|^p\,dm\right]^{1/p}, \qquad \text{for } 1\le p\prec\infty, \qquad (8)$$

and

$$d_p(A,B)=\ essential\ \sup_{x\in X}|\mu_A(x)-\mu_B(x)| \qquad \text{for } p=\infty, \qquad (9)$$

where $X\ne\phi$ is a Lebesgue measurable set, m is a Lebesgue measure on X. (See Klir and Yuan, 1995).

The membership functions of two fuzzy numbers are the same if the distance between them is zero, i.e.,

$$d_p(A,B)=0 \Rightarrow \mu_A(x)=\mu_B(x) \qquad \forall x \in (X-E),$$

If the two functions $d_1$and d_2 defined such that:

d_1 and d_2: $X_F \times X_F \to R^+$,

where X_F is a fuzzy set and $X=\{x_1,x_2,\ldots,x_n\}$ is a fuzzy random variable (FRV), and $A,B \in X_F$.

Then:

$$d_1(A,B)=\sum_{i=1}^{n}\left|\mu_A(x_i)-\mu_B(x_i)\right|, \tag{10}$$

and

$$d_2(A,B)=\sum_{i=1}^{n}\left(\mu_A(x_i)-\mu_B(x_i)\right)^2, \tag{11}$$

Are called fuzzy distances. (Rudin, 1984).

The FRVs used in this paper are considered as functions from a probability space (**Ω**,**A**,P) into the metric space ($F_c(R),d_\theta$), where θ>0. The sample mean $\overline{X}_n$ and sample variance $\sigma^2_{\theta,n}$ of the FRV X are defined by:

$$\overline{X}_n=\frac{1}{n}\left(X_1 \oplus X_2 \oplus \ldots \oplus X_n\right), \tag{12}$$

and

$$\sigma^2_{\theta,n}=\frac{1}{n}\sum_{i=1}^{n} d^2_\theta\left(X_i,\overline{X}_n\right). \tag{13}$$

If X and Y are two FRVs , then the Bertoluzza covariance between them is defined as:

$$\operatorname{cov}_\theta(X,Y)=\operatorname{cov}_{mid}(X,Y)+\theta \operatorname{cov}_{spr}(X,Y), \tag{14}$$

$$\operatorname{cov}_{mid}(X,Y)=\int_{[0,1]}\frac{1}{n}\sum_{i=1}^{n} mid\left[(X_i)_\alpha\right] mid\left[(Y_i)_\alpha\right] d\alpha-\int_{[0,1]} mid\left[(\overline{X}_n)_\alpha\right] mid\left[(\overline{Y}_n)_\alpha\right] d\alpha \tag{15}$$

$$\operatorname{cov}_{mid}(X,Y)=\int_{[0,1]}\frac{1}{n}\sum_{i=1}^{n} mid\left[(X_i)_\alpha\right] mid\left[(Y_i)_\alpha\right] d\alpha-\int_{[0,1]} mid\left[(\overline{X}_n)_\alpha\right] mid\left[(\overline{Y}_n)_\alpha\right] d\alpha$$

(3) Fuzzy Linear Regression Models

3.1 The Standard Linear Regression Models

Consider the following standard simple linear regression model:

$$Y_i=\beta_0+\beta_1 X_i+\varepsilon_i, \quad \text{i=1,2,…,n}, \tag{16}$$

where β_0, and β_1 are unknown parameters, X is the predictor, Y is the response variable and ε is the error term of the model, with $E(\varepsilon \setminus X)=0$ and finite variance. The least squares estimators of β_0, and β_1 are obtained by minimizing the sum of squared error criterion, Q, as follows:

$$Q=\arg\min_{\beta_0,\beta_1}\sum_{i=1}^{n}\left(Y_i-\beta_0-\beta_1 X_1\right)^2. \tag{17}$$

The resulting estimators denoted by b_0, and b_1 are as follows:

$$b_1 = \frac{\sum_{i=1}^{n}(x_i y_i) - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}, \qquad \text{and } b_0 = \bar{y} - b_1\bar{x}. \tag{18}$$

The multiple linear regression model is one:

$$Y = X\beta + \varepsilon\ , \tag{19}$$

where Y is an (n×1) column vector of the dependent variable, X is an (n×p) matrix of predictors, β is a (p×1) vector of unknown parameters to be estimated, and ε is an (n×1) vector of errors distributed as $N(0,\sigma^2 I_n)$. The least squares estimator of β , denoted by b is given by:

$$b = (X'X)^{-1} X'Y\ , \tag{20}$$

which is obtained by minimizing the corresponding criterion, Q as:

$$Q = \arg\min_{\beta}(Y - X\beta)'(Y - X\beta). \tag{21}$$

3.2 Simple Fuzzy Linear Regression Models

In the case of using fuzzy data, fuzzy regression models will be used to estimate the unknown parameters. Consider the following fuzzy simple linear regression models:

$$\tilde{y}_i = \beta_0 + \beta_1\tilde{x}_i + \tilde{\varepsilon}\ , \tag{22}$$

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 x_i + \tilde{\varepsilon}\ , \tag{23}$$

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1\tilde{x}_i + \tilde{\varepsilon}\ , \tag{24}$$

where $\beta_0, and \beta_1$, are crisp parameters, x is a crisp variable, $\tilde{\beta}_0, and \tilde{\beta}_1$ are fuzzy parameters, $\tilde{y}$ is a fuzzy response variable, $\tilde{x}$ is a fuzzy predictor. As a lack of linearity of $F_c(R^p)$, $\tilde{\varepsilon}$ is reduced to a non-FRV. (See Gonzalez-Rodriguez et al. (2009)).

The regression functions of models (22), (23), and (24) will be approximated as follows:

$$E(\tilde{Y} \setminus \tilde{X}) = \beta_0 + \beta_1\tilde{X}\ , \tag{25}$$

$$E(\tilde{Y} \setminus X) = \tilde{\beta}_0 + \tilde{\beta}_1 X\ , \tag{26}$$

$$E(\tilde{Y} \setminus \tilde{X}) = \tilde{\beta}_0 + \tilde{\beta}_1\tilde{X}\ , \tag{27}$$

The least squares estimators of the parameters in models (22):(24) are derived using using triangular and trapezoidal fuzzy numbers. The derivation is approximated by optimizing the least squares criterion. In this work, the least squares optimization criterion which is an extension version of that introduced by Diamond (1988) will be used.

3.3 The least Squares Approach for of the Simple Fuzzy Regression Models Using Triangular Fuzzy Numbers

The least squares estimators of the parameters in model (22) are obtained by minimizing the least squares criterion as follows:

$$Q(\beta_0, \beta_1) = \arg\min_{\beta_0,\beta_1} \sum_{i=1}^{n} d^2(\tilde{y}_i, \beta_0 + \beta_1\tilde{x}_i) \tag{28}$$

Diamond (1988) showed that there are two cases arising when $\beta_1 \geq 0$ or $\beta_1 \prec 0$. Using the triangular fuzzy number, the objective function in (28), when $\beta_1 \geq 0$, will be as follows:

$$
\begin{aligned}
Q^{+}(\beta_0,\beta_1) &= \underset{\beta_0,\beta_1}{\arg\min}\sum_{i=1}^{n} d^2\left(\tilde{y}_i,\beta_0+\beta_1\tilde{x}_i\right) \\
&= \underset{\beta_0,\beta_1}{\arg\min}\sum_{i=1}^{n}\left[\left(y_{il}-\beta_0-\beta_1 x_{il}\right)^2+\left(y_{im}-\beta_0-\beta_1 x_{im}\right)^2+\left(y_{ir}-\beta_0-\beta_1 x_{ir}\right)^2\right]
\end{aligned}
\tag{29}
$$

By differentiating of Eq. (29) with respect to the parameters β_1 and β_0, and equating the equations by zero:

$$
\frac{\partial Q^{+}(\beta_0,\beta_1)}{\partial\beta_1}=-2x_{i1l}\sum_{i=1}^{n}\left(y_{il}-\beta_0-\beta_1 x_{i1l}\right)-2x_{i1m}\sum_{i=1}^{n}\left(y_{im}-\beta_0-\beta_1 x_{i1m}\right)-2x_{i1r}\sum_{i=1}^{n}\left(y_{ir}-\beta_0-\beta_1 x_{i1r}\right)=0
$$

$$
\frac{\partial Q^{+}(\beta_0,\beta_1)}{\partial\beta_0}=-2\sum_{i=1}^{n}\left(y_{il}-\beta_0-\beta_1 x_{i1l}\right)-2\sum_{i=1}^{n}\left(y_{im}-\beta_0-\beta_1 x_{i1m}\right)-2\sum_{i=1}^{n}\left(y_{ir}-\beta_0-\beta_1 x_{i1r}\right)=0
$$

The least squares estimators, b_1^{+} and b_0^{+} of β_1 and β_0 respectively, are obtained as follows:

$$
b_1^{+}=\frac{\sum_{i=1}^{n}\left(x_{il}y_{il}+x_{im}y_{im}+x_{ir}y_{ir}\right)-3n\bar{x}\bar{y}}{\sum_{i=1}^{n}\left(x_{il}^2+x_{im}^2+x_{ir}^2\right)-3n\bar{x}^2},
\tag{30}
$$

$$
b_0^{+}=\bar{y}-b_1^{+}\bar{x},
\tag{31}
$$

where, y_{il}, y_{im}, and y_{ir} are the left, middle, and right value of y_i, respectively, for i=1,2,…,n. Also, x_{il}, x_{im}, and x_{ir} are the left, middle, and right value of x_i, respectively, for i=1,2,…,n. $\bar{y}=\sum_{i=1}^{n}\left(y_{il}+y_{im}+y_{ir}\right)/3n$, and $\bar{x}=\sum_{i=1}^{n}\left(x_{il}+x_{im}+x_{ir}\right)/3n$.

For the second case, where $\beta_1 \prec 0$, the objective function of (28) will be as follows:

$$
\begin{aligned}
Q^{-}(\beta_0,\beta_1) &= \underset{\beta_0,\beta_1}{\arg\min}\sum_{i=1}^{n} d^2\left(\tilde{y}_i,\beta_0+\beta_1\tilde{x}_i\right) \\
&= \underset{\beta_0,\beta_1}{\arg\min}\sum_{i=1}^{n}\left[\left(y_{il}-\beta_0-\beta_1 x_{ir}\right)^2+\left(y_{im}-\beta_0-\beta_1 x_{im}\right)^2+\left(y_{ir}-\beta_0-\beta_1 x_{il}\right)^2\right],
\end{aligned}
\tag{32}
$$

and differentiating of Eq. (32), the least squares estimators, b_1^{-} and b_0^{-} of β_1 and β_0 respectively, are obtained as follows:

$$
b_1^{-}=\frac{\sum_{i=1}^{n}\left(x_{il}y_{il}+x_{im}y_{im}+x_{ir}y_{ir}\right)-3n\bar{x}\bar{y}}{\sum_{i=1}^{n}\left(x_{il}^2+x_{im}^2+x_{ir}^2\right)-3n\bar{x}^2},
\tag{33}
$$

$$
b_0^{-}=\bar{y}-b_1^{-}\bar{x}.
\tag{34}
$$

Diamond (1988 [5], 1990[6]) showed that for every fuzzy nondegenerate data set that $b_1^{+}\geq b_1^{-}$, and the least squares estimators will be unique if the fuzzy nondegenerate data set is tight.

Definition (3.1)

Consider the fuzzy data sets $\tilde{y}_i = (y_{il}, y_{im}, y_{ir})$, and $\tilde{x}_i = (x_{il}, x_{im}, x_{ir})$, for i=1,2,…,n, the set is said to be nondegenerated, if not all observations in a set are made at the same datum.

Definition (3.2)

Consider the fuzzy data sets $\tilde{y}_i = (y_{il}, y_{im}, y_{ir})$, and $\tilde{x}_i = (x_{il}, x_{im}, x_{ir})$, for i=1,2,…,n, the set is said to be tight if either $b_1^+ \le 0$ or $b_1^- \ge 0$. If $b_1^- \ge 0$ the data set is said to be tight positive, and if $b_1^+ \le 0$ the data set is said to be tight negative. (Diamond (1988[5]).

The least squares estimators of the parameters in model (23) are obtained by minimizing the squared distances between the regression model and the regression function as follows:

$$Q(\tilde{\beta}_0, \tilde{\beta}_1) = \arg\min_{\beta_0, \beta_1} \sum_{i=1}^{n} d^2(\tilde{y}_i, \tilde{\beta}_0 + \tilde{\beta}_1 x_i) \qquad (35)$$

where $\tilde{\beta}_0 = (\beta_{0l}, \beta_{0m}, \beta_{0r})$ and $\tilde{\beta}_1 = (\beta_{1l}, \beta_{1m}, \beta_{1r})$ are two triangular fuzzy numbers.
Eq. (35) can be written as:

$$Q(\tilde{\beta}_0, \tilde{\beta}_1) = \arg\min_{\beta_0, \beta_1} \sum_{i=1}^{n} d^2(\tilde{y}_i, \tilde{\beta}_0 + \tilde{\beta}_1 x_i) = \arg\min_{\beta_0, \beta_1} \left[(y_{il} - \beta_{0l} - \beta_{1l} x_i)^2 + (y_{im} - \beta_{0m} - \beta_{1m} x_i)^2 + (y_{ir} - \beta_{0r} - \beta_{1r} x_i)^2 \right] \qquad (36)$$

By differentiating of Eq. (36) with respect to the parameters β_{1l}, β_{1m}, β_{1r} and β_{0l}, β_{0m}, β_{0r}, the least squares estimators, b_{1l}, b_{1m}, b_{1r} and b_{0l}, b_{0m}, b_{0r} are obtained when $x_i \ge 0$ as follows:

$$b_{1l} = \frac{\sum_{i=1}^{n} (x_i y_{il}) - n\bar{x}\bar{y}_l}{\sum_{i=1}^{n} (x_i^2) - n\bar{x}^2}, \quad b_{1m} = \frac{\sum_{i=1}^{n} (x_i y_{im}) - n\bar{x}\bar{y}_m}{\sum_{i=1}^{n} (x_i^2) - n\bar{x}^2}, \quad b_{1r} = \frac{\sum_{i=1}^{n} (x_i y_{ir}) - n\bar{x}\bar{y}_r}{\sum_{i=1}^{n} (x_i^2) - n\bar{x}^2}, \qquad (37)$$

$$b_{0l} = \bar{y}_l - b_{1l}\bar{x}, \qquad b_{0l} = \bar{y}_l - b_{1l}\bar{x} \qquad ,.\, b_{0r} = \bar{y}_r - b_{1r}\bar{x}. \qquad (38)$$

when $x_i < 0$, least squares estimators, b_{1l}, b_{1m}, b_{1r} and b_{0l}, b_{0m}, b_{0r} are obtained as follows:

$$b_{1l} = \frac{\sum_{i=1}^{n} (x_i y_{ir}) - n\bar{x}\bar{y}_r}{\sum_{i=1}^{n} (x_i^2) - n\bar{x}^2}, \quad b_{1m} = \frac{\sum_{i=1}^{n} (x_i y_{im}) - n\bar{x}\bar{y}_m}{\sum_{i=1}^{n} (x_i^2) - n\bar{x}^2}, \quad b_{1r} = \frac{\sum_{i=1}^{n} (x_i y_{il}) - n\bar{x}\bar{y}_l}{\sum_{i=1}^{n} (x_i^2) - n\bar{x}^2}, \qquad (37)$$

$$b_{0l} = \bar{y}_l - b_{1r}\bar{x}, \qquad b_{0m} = \bar{y}_m - b_{1m}\bar{x} \qquad , \qquad b_{0r} = \bar{y}_r - b_{1l}\bar{x}. \qquad (38)$$

The least squares estimators of the parameters in model (24) are obtained by minimizing the squared distances between the regression model and the regression function as follows:

$$Q(\tilde{\beta}_0, \tilde{\beta}_1) = \arg\min_{\beta_0, \beta_1} \sum_{i=1}^{n} d^2(\tilde{y}_i, \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i) \qquad (39)$$

where $\tilde{\beta}_0 = (\beta_{0l}, \beta_{0m}, \beta_{0r})$, $\tilde{\beta}_1 = (\beta_{1l}, \beta_{1m}, \beta_{1r})$, and $\tilde{x}_i = (x_{il}, x_{im}, x_{ir})$ are triangular fuzzy numbers, and $\tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i$ is approximately fuzzy number. (See Arabpour and Tata).
Eq. (39) can be written as:

$$Q\left(\tilde{\beta}_0,\tilde{\beta}_1\right)=\underset{\beta_0,\beta_1}{\arg\min}\sum_{i=1}^{n} d^2\left(\tilde{y}_i,\tilde{\beta}_0+\tilde{\beta}_1 x_i\right)=\underset{\beta_0,\beta_1}{\arg\min}\left[\left(y_{il}-\beta_{0l}-\beta_{1l}x_{il}\right)^2+\left(y_{im}-\beta_{0m}-\beta_{1m}x_{im}\right)^2+\left(y_{ir}-\beta_{0r}-\beta_{1r}x_{ir}\right)^2\right] \quad (40)$$

By differentiating of Eq. (40) with respect to the parameters β_{1l}, β_{1m}, β_{1r} and β_{0l}, β_{0m}, β_{0r}, the least squares estimators, b_{1l}, b_{1m}, b_{1r} and b_{0l}, b_{0m}, b_{0r} are obtained as follows when $\tilde{x}_i$'s and $\tilde{\beta}_1$ are positive fuzzy numbers.

$$b_{1l}=\frac{\sum_{i=1}^{n}\left(x_{il}y_{il}\right)-n\bar{x}_l\bar{y}_l}{\sum_{i=1}^{n}\left(x_{il}^2\right)-n\bar{x}_l^{\,2}},\; b_{1m}=\frac{\sum_{i=1}^{n}\left(x_{il}y_{im}\right)-n\bar{x}_m\bar{y}_m}{\sum_{i=1}^{n}\left(x_{im}^2\right)-n\bar{x}_m^{\,2}},\; b_{1r}=\frac{\sum_{i=1}^{n}\left(x_{ir}y_{ir}\right)-n\bar{x}_r\bar{y}_r}{\sum_{i=1}^{n}\left(x_{ir}^2\right)-n\bar{x}_r^{\,2}}, \quad (41)$$

$$b_{0l}=\bar{y}_l-b_{1r}\bar{x}_l,\qquad b_{0m}=\bar{y}_m-b_{1m}\bar{x}_m\qquad,\qquad b_{0r}=\bar{y}_r-b_{1l}\bar{x}_r. \quad (42)$$

The derivation of the fuzzy simple least squares estimators using trapezoidal fuzzy numbers can be easily found.

3.4 Multivariate Fuzzy Linear Regression Models

3.4.1 Multivariate Fuzzy Linear Regression Models for Fuzzy Predictors and Crisp Parameters

Consider the case of fuzzy simple linear regression models defined in (22), the multiple fuzzy regression model may be formalized as follows:

$$\tilde{y}_i=\beta_0+\beta_1\tilde{x}_{i1}+\beta_2\tilde{x}_{i2}+\ldots+\beta_p\tilde{x}_{ip}+\tilde{\varepsilon}_i. \quad (43)$$

Suppose using centered values of fuzzy predictors, Eq. (43) can be written in matrix form as follows:

$$\tilde{Y}=\tilde{X}\beta+\tilde{\varepsilon}, \quad (44)$$

where, $\tilde{Y}$ is an $(n\times 1)$ vector , $\tilde{X}$ is an $(n\times p)$ matrix of p fuzzy predictors, and β is a $(p\times 1)$ vector of unknown p crisp parameters. As a result of the lack of linearity of $F_c\left(R^p\right)$, $\tilde{\varepsilon}$ is reduced to a non-FRV ε. (See Gonzalez-Rodriguez et al. (2009)).

$\tilde{Y}$, $\tilde{X}$, β, and ε are formalized in matrix form as follows:

$$\tilde{Y}=\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \vdots\\ \tilde{y}_n\end{bmatrix},\tilde{X}=\begin{bmatrix}\tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1\times p}\\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2\times p}\\ \vdots & \vdots & \cdots & \vdots\\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{n\times p}\end{bmatrix},\;\beta=\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_p\end{bmatrix},\text{ and }\;\tilde{\varepsilon}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \vdots\\ \tilde{\varepsilon}_n\end{bmatrix},$$

where $\tilde{y}_i=\left(y_{il},y_{im},y_{ir}\right)$, and $\tilde{x}_{ij}=\left(x_{ijl},x_{ijm},x_{ijr}\right)$, for i=1,2,…,n, and j=1,2,…,p.

The least squares estimator of β in model (44), for triangular fuzzy variables, can be formalized as follows:

$$\hat{\beta}=\left(X_l'X_l+X_m'X_m+X_r'X_r\right)^{-1}\left[X_l'Y_l+X_m'Y_m+X_r'Y_r\right], \quad (45)$$

where,

$X_l=\left[x_{ijl}-\bar{x}_j\right]$, $X_m=\left[x_{ijm}-\bar{x}_j\right]$, $X_r=\left[x_{ijr}-\bar{x}_j\right]$, are $(n\times p)$ left, middle, and right fuzzy matrices of predictors. $Y_l=\left(y_{1l},y_{2l},\ldots,y_{nl}\right)$, $Y_m=\left(y_{1m},y_{2m},\ldots,y_{nm}\right)$, $Y_r=\left(y_{1r},y_{2r},\ldots,y_{nr}\right)$, are (n×1) response vectors such that:

$$y_{il}=x_{i1l}\beta_1+x_{i2l}\beta_2+\ldots+x_{ipl}\beta_p,\qquad \text{for i=1,2,…,n}$$

$$y_{im}=x_{i1m}\beta_1+x_{i2m}\beta_2+\ldots+x_{ipm}\beta_p,\qquad \text{for i=1,2,…,n}$$

$$y_{ir} = x_{i1r}\beta_1 + x_{i2r}\beta_2 + ... + x_{ipr}\beta_p, \qquad \text{for i=1,2,…,n}$$

The least squares estimator of β in model (44), for trapezoidal fuzzy variables, can be formalized as follows:

$$\hat{\beta} = \left(X_l'X_l + X_u'X_u + X_\upsilon'X_\upsilon + X_r'X_r\right)^{-1}\left[X_l'Y_l + X_u'Y_u + X_\upsilon'Y_\upsilon + X_r'Y_r\right], \qquad (46)$$

where,

$X_l = \left[x_{ijl} - \bar{x}_j\right]$, $X_u = \left[x_{iju} - \bar{x}_j\right]$, $X_\upsilon = \left[x_{ij\upsilon} - \bar{x}_j\right]$, $X_r = \left[x_{ijr} - \bar{x}_j\right]$, are ($n$×$p$) left, middle left, middle right, and right fuzzy matrices of predictors. $Y_l = \left(y_{1l}, y_{2l}, ..., y_{nl}\right)$, $Y_u = \left(y_{1u}, y_{2u}, ..., y_{nu}\right)$, $Y_\upsilon = \left(y_{1\upsilon}, y_{2\upsilon}, ..., y_{n\upsilon}\right)$, $Y_r = \left(y_{1r}, y_{2r}, ..., y_{nr}\right)$, are (n×1) response vectors such that:

$$y_{il} = x_{i1l}\beta_1 + x_{i2l}\beta_2 + ... + x_{ipl}\beta_p, \qquad \text{for i=1,2,…,n}$$
$$y_{iu} = x_{i1u}\beta_1 + x_{i2u}\beta_2 + ... + x_{ipu}\beta_p, \qquad \text{for i=1,2,…,n}$$
$$y_{i\upsilon} = x_{i1\upsilon}\beta_1 + x_{i2\upsilon}\beta_2 + ... + x_{ip\upsilon}\beta_p \qquad \text{for i=1,2,…,n}$$
$$y_{ir} = x_{i1r}\beta_1 + x_{i2r}\beta_2 + ... + x_{ipr}\beta_p, \qquad \text{for i=1,2,…,n}$$

3.4.2 Multivariate Fuzzy Linear Regression Models for Crisp Predictors and Fuzzy Parameters

Consider the case of fuzzy simple linear regression models defined in (23), the multiple fuzzy regression model can be generalized as follows:

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 x_{i1} + \tilde{\beta}_2 x_{i2} + ... + \tilde{\beta}_p x_{ip} + \varepsilon_i. \qquad (333)$$

Suppose using centered values of crisp predictors, Eq. (43) can be written in matrix form as follows:

$$\tilde{Y} = X\tilde{\beta} + \varepsilon, \qquad (44)$$

where, $\tilde{Y}$ is an (n×1) fuzzy vector , X is an (n×p) matrix of p crisp predictors, and $\tilde{\beta}$ is a (p×1) vector of unknown p fuzzy parameters. As a result of the lack of linearity of $F_c\left(R^p\right)$, $\tilde{\varepsilon}$ is reduced to a non-FRV ε. (See Gonzalez-Rodriguez et al. (2009)).

$\tilde{Y}$, X , $\tilde{\beta}$, and ε are formalized in matrix form as follows:

$$\tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_n \end{bmatrix}, X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1\times p} \\ x_{21} & x_{22} & \cdots & x_{2\times p} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n\times p} \end{bmatrix}, \tilde{\beta} = \begin{bmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_p \end{bmatrix}, \text{and } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix},$$

where $\tilde{y}_i = \left(y_{il}, y_{im}, y_{ir}\right)$, and $\tilde{\beta}_j = \left(\beta_{jl}, \beta_{jm}, \beta_{jr}\right)$, for i=1,2,…,n, and j=1,2,…,p.

The least squares estimator $\hat{\beta}$ of $\tilde{\beta}$ in model (44), for triangular fuzzy variables, can be formalized as follows:

$$\hat{\beta} = \left(\hat{\beta}_l, \hat{\beta}_m, \hat{\beta}_r\right),$$

where,

$$\hat{\beta}_l = \left(X'X\right)^{-1}\left[X'Y_l\right], \qquad (45)$$
$$\hat{\beta}_m = \left(X'X\right)^{-1}\left[X'Y_m\right],$$
$$\hat{\beta}_r = \left(X'X\right)^{-1}\left[X'Y_r\right],$$

where,

$X = \lfloor x_{ij} - \bar{x}_j \rfloor$, and $Y_l = (y_{1l}, y_{2l}, ..., y_{nl})$, $Y_m = (y_{1m}, y_{2m}, ..., y_{nm})$, $Y_r = (y_{1r}, y_{2r}, ..., y_{nr})$, are (n×1) response vectors such that:

$$y_{il} = x_{i1}\beta_{1l} + x_{i2}\beta_{2l} + ... + x_{ip}\beta_{pl}, \quad \text{for i=1,2,…,n}$$
$$y_{im} = x_{i1}\beta_{1m} + x_{i2}\beta_{2m} + ... + x_{ip}\beta_{pm}, \quad \text{for i=1,2,…,n}$$
$$y_{ir} = x_{i1}\beta_{1r} + x_{i2}\beta_{2r} + ... + x_{ip}\beta_{pr}, \quad \text{for i=1,2,…,n}$$

The least squares estimator of $\tilde{\beta}$ in model (44), for trapezoidal fuzzy variables, can be formalized as follows:

$$\hat{\beta} = (\hat{\beta}_l, \hat{\beta}_u, \hat{\beta}_v, \hat{\beta}_r),$$

where,

$$\hat{\beta}_l = (X'X)^{-1}[X'Y_l],$$
$$\hat{\beta}_u = (X'X)^{-1}[X'Y_u],$$
$$\hat{\beta}_m = (X'X)^{-1}[X'Y_v]$$
$$\hat{\beta}_r = (X'X)^{-1}[X'Y_r].$$

3.4.3 Multivariate Fuzzy Linear Regression Models for Fuzzy Predictors and Fuzzy Parameters

Consider the case of fuzzy simple linear regression models defined in (24), the multiple fuzzy regression model can be generalized as follows:

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1\tilde{x}_{i1} + \tilde{\beta}_2\tilde{x}_{i2} + ... + \tilde{\beta}_p\tilde{x}_{ip} + \varepsilon_i.$$

Suppose using centered values of crisp predictors, Eq. (43) can be written in matrix form as follows:

$$\tilde{Y} = \tilde{X}\tilde{\beta} + \varepsilon, \qquad (44)$$

where, $\tilde{Y}$ is an (*n*×*1*) fuzzy vector, $\tilde{X}$ is an (*n*×*p*) matrix of p fuzzy predictors, and $\tilde{\beta}$ is a (*p*×*1*) vector of unknown *p* fuzzy parameters. As a result of the lack of linearity of $F_c(R^p)$, $\tilde{\varepsilon}$ is reduced to a non-FRV ε. (See Gonzalez-Rodriguez et al. (2009)).

$\tilde{Y}$, $\tilde{X}$, $\tilde{\beta}$, and ε are formalized in matrix form as follows:

$$\tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_n \end{bmatrix}, \tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1\times p} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2\times p} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{n\times p} \end{bmatrix}, \tilde{\beta} = \begin{bmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_p \end{bmatrix}, \text{and } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix},$$

where $\tilde{y}_i = (y_{il}, y_{im}, y_{ir})$, $\tilde{x}_{ij} = (x_{ijl}, x_{ijm}, x_{ijr})$ and $\tilde{\beta}_j = (\beta_{jl}, \beta_{jm}, \beta_{jr})$, for i=1,2,…,n, and j=1,2,…,p.

The least squares estimator $\hat{\beta}$ of $\tilde{\beta}$ in model (44), for triangular fuzzy variables, can be formalized as follows:

$$\hat{\beta} = (\hat{\beta}_l, \hat{\beta}_m, \hat{\beta}_r),$$

where,

$$\hat{\beta}_l = (X_l'X_l)^{-1}[X_l'Y_l], \qquad (45)$$
$$\hat{\beta}_m = (X_m'X_m)^{-1}[X_m'Y_m],$$
$$\hat{\beta}_r = (X_r'X_r)^{-1}[X_r'Y_r],$$

where,

$X_l = \lfloor x_{ijl} - \bar{x}_j \rfloor$, $X_m = \lfloor x_{ijm} - \bar{x}_j \rfloor$, $X_r = \lfloor x_{ijr} - \bar{x}_j \rfloor$, are ($n \times p$) left, middle, and right fuzzy matrices of predictors. $Y_l = (y_{1l}, y_{2l}, ..., y_{nl})$, $Y_m = (y_{1m}, y_{2m}, ..., y_{nm})$, $Y_r = (y_{1r}, y_{2r}, ..., y_{nr})$, are (n×1) response vectors such that:

$$y_{il} = x_{i1l}\beta_{1l} + x_{i2l}\beta_{2l} + ... + x_{ipl}\beta_{pl}, \qquad \text{for i=1,2,…,n}$$

$$y_{im} = x_{i1m}\beta_{1m} + x_{i2m}\beta_{2m} + ... + x_{ipm}\beta_{pm}, \qquad \text{for i=1,2,…,n}$$

$$y_{ir} = x_{i1r}\beta_{1r} + x_{i2r}\beta_{2r} + ... + x_{ipr}\beta_{pr}, \qquad \text{for i=1,2,…,n}$$

The least squares estimator of $\tilde{\beta}$ in model (44), for trapezoidal fuzzy variables, can be formalized as follows:

$$\hat{\beta}_l = (X_l' X_l)^{-1}[X_l' Y_l],$$

$$\hat{\beta}_u = (X_u' X_u)^{-1}[X_u' Y_u],$$

$$\hat{\beta}_v = (X_v' X_v)^{-1}[X_v' Y_v]$$

$$\hat{\beta}_r = (X_r' X_r)^{-1}[X_r' Y_r].$$

(4) The Proposed Mixed Fuzzy Crisp (MFC) Regression Model

All the fuzzy multiple regression models that have been considered in the literature handled the cases where all the predictors are fuzzy or all are crisp.

In this section, a new multiple linear regression model which mixes the fuzzy and crisp predictors in one model called "Mixed Fuzzy Crisp" (MFC) regression model, is proposed.The least squares approach for the new model is derived based on positive tight data as defined in (3.2) and triangular fuzzy numbers. Also, the properties of the resulting regression parameters are introduced in two cases: first, when the parameters are fuzzy, and second when the parameters are crisp.

4.1 The Proposed Mixed Fuzzy Crisp (MFC) Regression Model Using Crisp Parameters

Consider the case where the multiple linear regression model concludes some fuzzy and some crisp predictors. The computations will be done using triangular fuzzy number, and can applied to trapezoidal one. Assuming centered predictors, the proposed simplest form of multiple model that contain two predictors, one is crisp and the other is fuzzy, with crisp parameters will be as follows:

$$\tilde{y}_i = \beta_1 \tilde{x}_{i1} + \beta_2 x_{i2} + \varepsilon_i . \qquad (47)$$

where $\tilde{y}_i = (y_{il}, y_{im}, y_{ir})$, and $\tilde{x}_{i1} = (x_{i1l}, x_{i1m}, x_{i1r})$, for i=1,2,…,n, $x_{i2} = (x_{im}, x_{im}, x_{im})$, and ε_i is a non-fuzzy error with mean equal zero. The regression function of model (47) will be as follows:

$$E(\tilde{y} \setminus \tilde{x}_1, x_2) = \beta_1 \tilde{x}_1 + \beta_2 x_2 .$$

The derivation of the least squares estimators is done by minimizing the squared distances between the regression model and the regression function as follows:

$$Q(\beta_1, \beta_2) = \underset{\beta_0, \beta_1}{\arg\min} \sum_{i=1}^{n} d^2(\tilde{y}_i, \beta_1 \tilde{x}_{i1} + \beta_2 x_{i2}) = \underset{\beta_0, \beta_1}{\arg\min} \sum_{i=1}^{n} (\tilde{y}_i, \beta_1 \tilde{x}_{i1} + \beta_2 x_{i2})^2$$

$$= \underset{\beta_0, \beta_1}{\arg\min} \left[\sum_{i=1}^{n} (\tilde{y}_{il} - \beta_1 x_{i1l} - \beta_1 x_{i2})^2 + \sum_{i=1}^{n} (\tilde{y}_{im} - \beta_1 x_{i1m} - \beta_2 x_{i2})^2 + \sum_{i=1}^{n} (\tilde{y}_{ir} - \beta_1 x_{i1r} - \beta_1 x_{i2})^2 \right] \qquad (48)$$

By differentiating of Eq. (48) with respect to the parameters β_1, and β_2, the following equations are obtained:

$$\frac{\partial Q^{+}(\beta_0,\beta_1)}{\partial \beta_1} = -2x_{i1l}\sum_{i=1}^{n}(y_{il}-\beta_1 x_{i1l}-\beta_2 x_{i2}) - 2x_{i1m}\sum_{i=1}^{n}(y_{im}-\beta_1 x_{i1m}-\beta_2 x_{i2}) - 2x_{i1r}\sum_{i=1}^{n}(y_{ir}-\beta_1 x_{i1r}-\beta_2 x_{i2}) = 0$$

$$\therefore \sum_{i=1}^{n} x_{i1l}(y_{il}-\beta_1 x_{i1l}-\beta_2 x_{i2}) + x_{i1m}(y_{im}-\beta_1 x_{i1m}-\beta_2 x_{i2}) + x_{i1r}(y_{ir}-\beta_1 x_{i1r}-\beta_2 x_{i2}) = 0$$

$$\therefore \beta_1\sum_{i=1}^{n}x_{i1l}^2 + \beta_2\sum_{i=1}^{n}x_{i1l}x_2 + \beta_1\sum_{i=1}^{n}x_{i1m}^2 + \beta_2\sum_{i=1}^{n}x_{i1m}x_2 + \beta_1\sum_{i=1}^{n}x_{i1r}^2 + \beta_2\sum_{i=1}^{n}x_{i1r}x_2 = \sum_{i=1}^{n}x_{i1l}y_{il} + \sum_{i=1}^{n}x_{i1m}y_{im} + \sum_{i=1}^{n}x_{i1r}y_{ir}$$

$$\therefore \beta_1\sum_{i=1}^{n}\left(x_{i1l}^2 + x_{i1m}^2 + x_{i1r}^2\right) + \beta_2\left(\sum_{i=1}^{n}x_{i1l}x_2 + \sum_{i=1}^{n}x_{i1m}x_2 + \sum_{i=1}^{n}x_{i1r}x_2\right) = \sum_{i=1}^{n}x_{i1l}y_{il} + \sum_{i=1}^{n}x_{i1m}y_{im} + \sum_{i=1}^{n}x_{i1r}y_{ir}\ ,(49)$$

and,

$$\frac{\partial Q^{+}(\beta_0,\beta_1)}{\partial \beta_2} = -2x_{i2}\sum_{i=1}^{n}(y_{il}-\beta_1 x_{i1l}-\beta_2 x_{i2}) - 2x_{i2}\sum_{i=1}^{n}(y_{im}-\beta_1 x_{i1m}-\beta_2 x_{i2}) - 2x_{i2}\sum_{i=1}^{n}(y_{ir}-\beta_1 x_{i1r}-\beta_2 x_{i2}) = 0$$

$$\therefore \sum_{i=1}^{n}x_{i2}(y_{il}-\beta_1 x_{i1l}-\beta_2 x_{i2}) + \sum_{i=1}^{n}x_{i2}(y_{im}-\beta_1 x_{i1m}-\beta_2 x_{i2}) + \sum_{i=1}^{n}x_{i2}(y_{ir}-\beta_1 x_{i1r}-\beta_2 x_{i2}) = 0$$

$$\therefore \beta_1\sum_{i=1}^{n}x_{i1l}x_{i2} + \beta_1\sum_{i=1}^{n}x_{i1m}x_{i2} + \beta_1\sum_{i=1}^{n}x_{i1r}x_{i2} + 3\beta_2\sum_{i=1}^{n}x_{i2}^2 = \sum_{i=1}^{n}x_{i2}y_{il} + \sum_{i=1}^{n}x_{i2}y_{im} + \sum_{i=1}^{n}x_{i2}y_{ir}\ . \qquad (50)$$

Solving the equations (49) and (50), the least squares estimators, $\hat{\beta}_1$, and $\hat{\beta}_2$, of β_1, and β_2 are obtained respectively, as follows:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_{i1l}y_{il} + x_{i1m}y_{im} + x_{i1r}y_{ir}) - 3\bar{x}_1\bar{y}\sum_{i=1}^{n}(x_{i2})}{\sum_{i=1}^{n}\left(x_{il}^2 + x_{im}^2 + x_{ir}^2\right) - 3\bar{x}_1^2\sum_{i=1}^{n}(x_{i2})}, \qquad (51)$$

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n}(x_{i1l}y_{il} + x_{i1m}y_{im} + x_{i1r}y_{ir}) - \hat{\beta}_1\sum_{i=1}^{n}\left(x_{il}^2 + x_{im}^2 + x_{ir}^2\right)}{\bar{x}_1\sum_{i=1}^{n}(x_{i2})}, \qquad (52)$$

where, y_{il}, y_{im}, and y_{ir} are the left, middle, and right value of y_i, respectively, for i=1,2,…,n. Also, x_{i1l}, x_{i1m}, and x_{i1r} are the left, middle, and right i's value of $\tilde{x}_1$, respectively, for i=1,2,…,n. $\bar{y} = \sum_{i=1}^{n}(y_{il}x_{i2} + y_{im}x_{i2} + y_{ir}x_{i2}) / \sum_{i=1}^{n}x_{i2}$, and $\bar{x}_1 = \sum_{i=1}^{n}(x_{il} + x_{im} + x_{ir}) / \sum_{i=1}^{n}x_{i2}$ are the weighted means of $\tilde{y}$ and $\tilde{x}_1$, respectively, using the observations of the crisp predictor x_2 as weights. All the above results can be shown for trapezoidal fuzzy data.

4.2 The Proposed Mixed Fuzzy Crisp (MFC) Regression Model Using Fuzzy Parameters

Suppose in model (47) that both the parameters β_1 and β_2 are triangular fuzzy numbers, the MFC model will be defined as follows:

$$\tilde{y}_i = \tilde{\beta}_1 \tilde{x}_{i1} + \tilde{\beta}_2 x_{i2} + \varepsilon_i . \tag{53}$$

where $\tilde{\beta}_1 = (\beta_{1l}, \beta_{1m}, \beta_{1r})$, $\tilde{\beta}_2 = (\beta_{2l}, \beta_{2m}, \beta_{2r})$, $\tilde{y}_i = (y_{il}, y_{im}, y_{ir})$, and $\tilde{x}_{i1} = (x_{i1l}, x_{i1m}, x_{i1r})$, for i=1,2,…,n, $x_{i2} = (x_{im}, x_{im}, x_{im})$, and ε_i is a non-fuzzy error with mean equal zero. The regression function of model (52) will be as follows:

$$E(\tilde{y} \setminus \tilde{x}_1, x_2) = \tilde{\beta}_1 \tilde{x}_1 + \tilde{\beta}_2 x_2 .$$

The derivation of the least squares estimators is done by minimizing the squared distances between the regression model and the regression function as follows:

$$\begin{aligned} Q(\beta_1, \beta_2) &= \arg\min_{\tilde{\beta}_1, \tilde{\beta}_2} \sum_{i=1}^{n} d^2\left(\tilde{y}_i, \tilde{\beta}_1 \tilde{x}_{i1} + \tilde{\beta}_2 x_{i2}\right) = \arg\min_{\tilde{\beta}_1, \tilde{\beta}_2} \sum_{i=1}^{n} \left(\tilde{y}_i, \tilde{\beta}_1 \tilde{x}_{i1} + \tilde{\beta}_2 x_{i2}\right)^2 \\ &= \arg\min_{\tilde{\beta}_1, \tilde{\beta}_2} \left[\sum_{i=1}^{n} (\tilde{y}_{il} - \beta_{1l} x_{i1l} - \beta_{2l} x_{i2})^2 + \sum_{i=1}^{n} (\tilde{y}_{im} - \beta_{1m} x_{i1m} - \beta_{2m} x_{i2})^2 + \sum_{i=1}^{n} (\tilde{y}_{ir} - \beta_{1r} x_{i1r} - \beta_{2r} x_{i2})^2 \right] \end{aligned} \tag{54}$$

By differentiating of Eq. (54) with respect to the parameters $\beta_{1l}, \beta_{1m}, \beta_{1r}$, and $\beta_{2l}, \beta_{2m}, \beta_{2r}$, then equating the resulting outputs to zero, the least squares estimators, $\hat{\beta}_{1l}, \hat{\beta}_{1m}, \hat{\beta}_{1r}$ and $\hat{\beta}_{2l}, \hat{\beta}_{2m}, \hat{\beta}_{2r}$ are obtained as follows:

$$\hat{\beta}_{1l} = \frac{\sum_{i=1}^{n}(x_{i1l} y_{il}) - \bar{x}_{1l} \bar{y}_l \sum_{i=1}^{n}(x_{i2})}{\sum_{i=1}^{n}(x_{i1l}^2) - \bar{x}_{1l}^2 \sum_{i=1}^{n}(x_{i2})}, \quad \hat{\beta}_{1m} = \frac{\sum_{i=1}^{n}(x_{i1m} y_{im}) - \bar{x}_{1m} \bar{y}_m \sum_{i=1}^{n}(x_{i2})}{\sum_{i=1}^{n}(x_{i1m}^2) - \bar{x}_{1m}^2 \sum_{i=1}^{n}(x_{i2})}, \quad \hat{\beta}_{1r} = \frac{\sum_{i=1}^{n}(x_{i1r} y_{ir}) - \bar{x}_{1r} \bar{y}_r \sum_{i=1}^{n}(x_{i2})}{\sum_{i=1}^{n}(x_{i1r}^2) - \bar{x}_{1r}^2 \sum_{i=1}^{n}(x_{i2})}, \tag{55}$$

$$\hat{\beta}_{2l} = \frac{\sum_{i=1}^{n}(x_{i1l} y_{il}) - \hat{\beta}_{1l} \sum_{i=1}^{n}(x_{i1l}^2)}{\bar{x}_{1l} \sum_{i=1}^{n}(x_{i2})}, \quad \hat{\beta}_{2m} = \frac{\sum_{i=1}^{n}(x_{i1m} y_{im}) - \hat{\beta}_{1m} \sum_{i=1}^{n}(x_{i1m}^2)}{\bar{x}_{1m} \sum_{i=1}^{n}(x_{i2})}, \quad \hat{\beta}_{2l} = \frac{\sum_{i=1}^{n}(x_{i1r} y_{ir}) - \hat{\beta}_{1r} \sum_{i=1}^{n}(x_{i1r}^2)}{\bar{x}_{1r} \sum_{i=1}^{n}(x_{i2})}, \tag{56}$$

where, y_{il}, y_{im}, and y_{ir} are the left, middle, and right value of y_i, respectively, for i=1,2,…,n. Also, x_{i1l}, x_{i1m}, and x_{i1r} are the left, middle, and right i's value of $\tilde{x}_1$, respectively, for i=1,2,…,n.

Using the observations of the crisp predictor x_2 as weight, $\bar{y}_l = \sum_{i=1}^{n}(y_{il} x_{i2}) / \sum_{i=1}^{n} x_{i2}$, $\bar{y}_m = \sum_{i=1}^{n}(y_{im} x_{i2}) / \sum_{i=1}^{n} x_{i2}$, $\bar{y}_r = \sum_{i=1}^{n}(y_{ir} x_{i2}) / \sum_{i=1}^{n} x_{i2}$ are the weighted means of y_l, y_m, and y_r respectively. Also, $\bar{x}_{1l} = \sum_{i=1}^{n}(x_{i1l}) / \sum_{i=1}^{n} x_{i2}$, $\bar{x}_{1m} = \sum_{i=1}^{n}(x_{i1m}) / \sum_{i=1}^{n} x_{i2}$, $\bar{x}_{1r} = \sum_{i=1}^{n}(x_{i1r}) / \sum_{i=1}^{n} x_{i2}$ are the weighted means of x_{1l}, x_{1m}, and x_{1r}, respectively. All the above results can be shown for trapezoidal fuzzy data.

(5) A Simulation Study

To illustrate the effectiveness of the proposed MFC regression model, a simulation study is conducted to compare the performance of MFC regression model with MF regression one. Two groups of models are introduced with two predictors, in the first group MFC and MF models with crisp parameters are used, and in the second group MFC and MF models with fuzzy parameters are considered as follows:

5.1 First Group

Model (1) MFC regression model: $\tilde{y}_i = \beta_1\tilde{x}_{i1} + \beta_2 x_{i2} + \varepsilon_i$, for i=1,2,…,n

with the following left, center, and right models:

$$y_{il} = x_{i1l}\beta_1 + x_{i2}\beta_2 , \qquad \text{for } i=1,2,\ldots,n$$
$$y_{im} = x_{i1m}\beta_1 + x_{i2}\beta_2 , \qquad \text{for } i=1,2,\ldots,n$$
$$y_{ir} = x_{i1r}\beta_1 + x_{i2}\beta_2 , \qquad \text{for } i=1,2,\ldots,n$$

Model (2) MF regression model: $\tilde{y}_i = \beta_1\tilde{x}_{i1} + \beta_2\tilde{x}_{i2} + \varepsilon_i$,

with the following left, center, and right sub-models:

$$y_{il} = x_{i1l}\beta_1 + x_{i2l}\beta_2 , \qquad \text{for } i=1,2,\ldots,n$$
$$y_{im} = x_{i1m}\beta_1 + x_{i2m}\beta_2 , \qquad \text{for } i=1,2,\ldots,n$$
$$y_{ir} = x_{i1r}\beta_1 + x_{i2r}\beta_2 , \qquad \text{for } i=1,2,\ldots,n$$

The triangular data set of $\tilde{x}_{i1} = (x_{i1l}, x_{i1m}, x_{i1r})$ and $\tilde{x}_{i2} = (x_{i2l}, x_{i2m}, x_{i2r})$ are generated from the normal distribution, and repeated 100 times, as follows:

x_{1l} □*N(0.5,2),*

x_{1m} □*N(1,2),*

x_{1r} □*N(2,4).*

The error term is supposed to distribute as normal with mean zero and variance one, i.e., ε □*N(0,1),* β_1=0.5 and β_2 =1.5.

The criterion used to compare the model (1) and model (2) is $\tilde{R}^2$, which is defined as:

$$\tilde{R}^2 = 1 - \frac{d^2(\tilde{y}, \hat{y})}{d^2(\tilde{y}, \bar{y})}, \qquad (57)$$

where, $d^2(\tilde{y}, \hat{y})$ is the squared distance between $\tilde{y} = (y_l, y_c, y_r)$ and $\hat{y} = (\hat{y}_l, \hat{y}_c, \hat{y}_r)$. Also, $d^2(\tilde{y}, \bar{y})$ is the squared distance between $\tilde{y} = (y_l, y_c, y_r)$ and $\bar{y} = (\bar{y}_l, \bar{y}_c, \bar{y}_r)$.

In Table (1), the multiple fuzzy model (MF) and mixed fuzzy crisp model (MFC) are compared using $\tilde{R}^2$ criterion as defined in (57). Best results are obtained for the MFC model in the form of greater values of the left $\tilde{R}^2$ compared to the left MF for all sample sizes. The improve of the right $\tilde{R}^2$ is noted for small sample sizes (n=5). Generally, the higher values of $\tilde{R}^2$ are obtained for smaller sample sizes of the two models MF and MFC. These results prove the validity of the fuzzy regression for vague and small data.

Table (1): $\tilde{R}^2$ (left, center, right) for the multiple fuzzy (MF) regression model, and the proposed mixed fuzzy crisp (MFC) regression model with different sample sizes, n=5,10,20,50,100,200, β_1=0.5 and β_2 =1.5.

n=5	Model	Left	Center	Right	n=50	Model	Left	Center	Right
	MF	0.9349	0.9496	0.9581		MF	0.9079	0.9415	0.9826
	MFC	0.9703	0.9496	0.9895		MFC	0.9567	0.9415	0.9342
n=10	**Model**	**Left**	**Center**	**Right**	**n=100**	**Model**	**Left**	**Center**	**Right**
	MF	0.9634	0.9936	0.9927		MF	0.7296	0.9074	0.9733
	MFC	0.9899	0.9936	0.9896		MFC	0.9068	0.9074	0.9363
n=20	**Model**	**Left**	**Center**	**Right**	**n=200**	**Model**	**Left**	**Center**	**Right**
	MF	0.8489	0.9463	0.9771		MF	0.8052	0.9201	0.9788
	MFC	0.9548	0.9463	0.9497		MFC	0.9236	0.9201	0.9409

5.2 Second Group

Model (1) MFC regression model: $\tilde{y}_i = \tilde{\beta}_1 \tilde{x}_{i1} + \tilde{\beta}_2 x_{i2} + \varepsilon_i$, for i=1,2,…,n

with the following left, center, and right models:

$$y_{il} = x_{i1l}\beta_{1l} + x_{i2}\beta_{2l}, \qquad \text{for } i=1,2,\ldots,n$$

$$y_{im} = x_{i1m}\beta_{1m} + x_{i2}\beta_{2m}, \qquad \text{for } i=1,2,\ldots,n$$

$$y_{ir} = x_{i1r}\beta_{1r} + x_{i2}\beta_{2r}, \qquad \text{for } i=1,2,\ldots,n$$

Model (2) MF regression model: $\tilde{y}_i = \tilde{\beta}_1 \tilde{x}_{i1} + \tilde{\beta}_2 \tilde{x}_{i2} + \varepsilon_i$

with the following left, center, and right models:

$$y_{il} = x_{i1l}\beta_{1l} + x_{i2l}\beta_{2l}, \qquad \text{for } i=1,2,\ldots,n$$

$$y_{im} = x_{i1m}\beta_{1m} + x_{i2m}\beta_{2m}, \qquad \text{for } i=1,2,\ldots,n$$

$$y_{ir} = x_{i1r}\beta_{1r} + x_{i2r}\beta_{2r}, \qquad \text{for } i=1,2,\ldots,n$$

The triangular data set of $\tilde{x}_{i1} = (x_{i1l}, x_{i1m}, x_{i1r})$ and $\tilde{x}_{i2} = (x_{i2l}, x_{i2m}, x_{i2r})$ are generated from the normal distribution, and repeated 100 times, as follows:

$x_{1l} \sim N(0.5,2)$,

$x_{1m} \sim N(1,2)$,

$x_{1r} \sim N(2,4)$.

The error term is supposed to distribute as normal with mean zero and variance one, i.e., $\varepsilon \sim N(0,1)$, $\tilde{\beta}_1 = (0.5,1.0,1.5)$ and $\tilde{\beta}_2 = (0.5,1.0,1.5)$. The criterion $\tilde{R}^2$ is used to compare the MFC and MF regression models.

In Table (2), as in the first group, it is found that best results are obtained for the MFC model in the form of greater values of the left $\tilde{R}^2$ compared to the left MF for all sample sizes. The improve of the right $\tilde{R}^2$ is noted for small sample sizes (n=5). Generally, the higher values of $\tilde{R}^2$ are obtained for smaller sample sizes for the two models MF and MFC. These results prove the validity of the fuzzy regression for small data.

Table (2): $\tilde{R}^2$ (left, center, right) for the multiple fuzzy (MF) regression model, and the proposed mixed fuzzy crisp (MFC) regression model with different sample sizes, n=5,10,20,50,100,200, $\tilde{\beta}_1 = (0.5,1.0,1.5)$ and $\tilde{\beta}_2 = (0.5,1.0,1.5)$.

n=5	Model	Left	Center	Right	n=50	Model	Left	Center	Right
	MF	0.7343	0.8700	0.9942		MF	0.8233	0.9218	0.9868
	MFC	0.8366	0.8700	0.9979		MFC	0.8757	0.9218	0.9742
n=10	**Model**	**Left**	**Center**	**Right**	**n=100**	**Model**	**Left**	**Center**	**Right**
	MF	0.9006	0.9893	0.9947		MF	0.3830	0.8864	0.9842
	MFC	0.9421	0.9893	0.9936		MFC	0.5826	0.8864	0.9815
n=20	**Model**	**Left**	**Center**	**Right**	**n=200**	**Model**	**Left**	**Center**	**Right**
	MF	0.6505	0.9533	0.9910		MF	0.6378	0.9083	0.9884
	MFC	0.8399	0.9533	0.9887		MFC	0.7392	0.9083	0.9834

(6) Conclusions

In this paper the simple linear regression model is extended to the multiple one and estimated with the least squares approach. This extension is based on adding both fuzzy and crisp predictors to the linear regression model, and the resulting model is called the mixed fuzzy crisp (MFC). Our extended model is evaluated using the extended fuzzy $\tilde{R}^2$. Simulated data examples are applied to compare the results of MFC model with the multiple fuzzy (MF) regression model using triangular fuzzy numbers. Best results are obtained in the form of larger values of $\tilde{R}^2$ of MFC compared to MF especially for small sample sizes. These results support using MFC model for small data size and for large size of tight data.

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