American Review of Mathematics and Statistics
December 2018, Vol. 6, No. 2, pp. 1-8
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v6n2a1
URL: https://doi.org/10.15640/arms.v6n2a1

A Variable Structural Control for a Flexible Plate

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Abstract

A variable structural control problem of a flexible thin plate formulated by partial differential equations with viscoelastic boundary conditions is studied in this paper. The problem is written in standard form of linear infinite dimensional system in an appropriate energy Hilbert space. The semi group approach of linear operators is adopted in investigating well-posedness of the closed loop system. A variable structural control for the system is proposed, and meanwhile an equivalent control method is applied to the thin plate system so that the thin plate can be exponentially stable and the actual sliding mode can be approximated by ideal sliding mode in any accuracy in terms of semi group approach.

Keywords: Partial differential equations, Flexible plate, Variable structural control, Semigroup of linear operators

AMS Subject Classification: 35B35, 93C20

1 Introduction

The problems of elastic structures with viscoelastic boundary conditions have been studied extensively by many articles (see References [1]-[5]). Motivated by the work on wave and heat equations mentioned above, in this article we are concerned with an elastic thin plate which occupies a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary Γ. Assume that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0 and Γ_1 are relatively open subsets of $\Gamma, \Gamma_0 \neq \emptyset$ has positive boundary measure, and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. If Γ_0 is clamped and the memory effect on Γ_1 is taken into account, the vertical deflection $y(x,t)$ of the thin elastic plate satisfies the following partial differential equation:

$$y_{tt}(x,t) + \Delta^2 y(x,t) = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \qquad (1.1a)$$
$$y(x,t) = \partial_v y(x,t) = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \qquad (1.1b)$$
$$\mathcal{B}_1 y(x,t) - \int_0^\infty g'(s)\, \partial_v [y(x,t) - y(x,t-s)] \mathrm{d}s = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \qquad (1.1c)$$
$$\mathcal{B}_2 y(x,t) + \int_0^\infty g'(s) [y(x,t) - y(x,t-s)] \mathrm{d}s = u(x,t), \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \qquad (1.1d)$$
$$y(x,0^+) = y_0(x), \quad y_t(x,0^+) = y_1(x), \qquad (1.1e)$$
$$y(x,-s) = \vartheta(x,t), \quad for \quad 0 < s < \infty, \qquad (1.1f)$$

where g is the relaxation function, u is the boundary control, y_0, y_1, ϑ are the given initial conditions. $\mathcal{B}_1, \mathcal{B}_2$ are the following boundary operators:

$$\mathcal{B}_1 y = \Delta_y + (1-\mu)(2v_1 v_2 \frac{\partial^2 y}{\partial x_1 \partial x_2} - v_1^2 \frac{\partial^2 y}{\partial x_2^2} - v_2^2 \frac{\partial^2 y}{\partial x_1^2}),$$
$$\mathcal{B}_2 y = \partial_v \Delta_y + (1-\mu)\, \partial_\tau [(v_1^2 - v_2^2) \frac{\partial^2 y}{\partial x_1 \partial x_2} + v_1 v_2 (\frac{\partial^2 y}{\partial x_2^2} - \frac{\partial^2 y}{\partial x_1^2})],$$

$v = (v_1, v_2)$ is the unit outer normal vector, $\tau = (-v_2, v_1)$ is the unit tangent vector, and $0 < \mu < \frac{1}{2}$ is the Poisson ratio.

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Throughout the article, we assume always that the function $g(\cdot)$ satisfies the following conditions:

(g_1) $g(\cdot) \in C^2[0,\infty)$;
(g_2) $g(t) > 0$, $g'(t) < 0$, $g''(t) \geq 0$ for $t \geq 0$;
(g_3) $g(\infty) > 0$;
$(g_4) g'(t) \geq -kg''(t)$ for some $k > 0$ and all $t \geq 0$.

Condition $(g2)$ implies that the memory of the boundary is strictly decreasing and the rate of memory loss is also decreasing. From $(g2)$, we have also that both $g(\infty)$ and $g'(\infty)$ exist, $g'(\infty) \geq 0$. Condition $(g3)$ means that the material behaves like an elastic solid at $t = \infty$. Condition $(g4)$ implies that $g'(t)$ decays exponentially, in particular, $g'(\infty) = 0$.

The energy corresponding to the system (1) is defined by

$$\begin{aligned} E(t) \quad = \quad & \frac{1}{2}a(y(\cdot,t)) + \int_\Omega |y_t(x,t)|^2 \mathrm{d}x - \int_0^\infty \int_{\Gamma_1} g'(s)[|\partial_v(y(x,t) - y(x,t-s))|^2 \\ & + |y(x,t) - y(x,t-s)|^2] \mathrm{d}\Gamma \mathrm{d}s \end{aligned} \tag{1.2}$$

where $a(w) = a(w,w)$ and

$$\begin{aligned} a(w_1,w_2) \quad = \quad & \int_\Omega [\frac{\partial^2 w_1}{\partial x_1^2}\overline{\frac{\partial^2 w_2}{\partial x_1^2}}] + \frac{\partial^2 w_1}{\partial x_2^2}\overline{\frac{\partial^2 w_2}{\partial x_2^2}} + \mu(\frac{\partial^2 w_1}{\partial x_1^2}\overline{\frac{\partial^2 w_2}{\partial x_2^2}}] + \frac{\partial^2 w_1}{\partial x_2^2}\overline{\frac{\partial^2 w_2}{\partial x_1^2}}) \\ & + 2(1-\mu)\frac{\partial^2 w_1}{\partial x_1 \partial x_2}\overline{\frac{\partial^2 w_2}{\partial x_1 \partial x_2}}]\mathrm{d}x, \quad \forall w_1, w_2 \in H^2(\Omega). \end{aligned} \tag{1.3}$$

2. Well-Posedness of the System with Feedback Control

In this section, we shall formulate the system (1.1a-1.1f) into a standard linear infinite dimensional space with a output feedback control. Let

$$W = \{w \in H^2(\Omega) | w|_{\Gamma_0} = \partial_v w|_{\Gamma_0} = 0\},$$
$$\| w \|_W^2 = a(w), \quad \forall w \in W,$$

and define the "boundary memory space" by

$$Z = L^2(0,\infty; |g'(.)|; H^1(\Gamma_1)),$$
$$\| z \|_Z^2 = \int_0^\infty |g'(s)| \left[\| \partial_v z(s) \|_{L^2(\Gamma_1)}^2 + \| z(s) \|_{L^2(\Gamma_1)}^2 \right] \mathrm{d}s, \qquad \forall z \in Z.$$

Set

$$\mathcal{H} = W \times L^2(\Omega) \times Z$$

equipped with the inner product induced norm

$$\| (w,v,z) \|_{\mathcal{H}}^2 = \| w \|_W^2 + \| v \|_{L^2_\Omega}^2 + \| z \|_Z^2, \quad \forall (w,v,z) \in \mathcal{H}.$$

It is easy to see that $\mathcal{H}$ is a Hilbert space.

Remark We have that $a(\cdot)^{\frac{1}{2}}$ is an equivalent norm on W since $\Gamma_0 \neq \emptyset$ has positive boundary measure. Moreover, it is obvious that $(\| \partial_v z \|_{L^2(\Gamma_1)}^2 + \| z^2 \|_{L^2(\Gamma_1)}^2)^{\frac{1}{2}}$ is an equivalent norm on $H^1(\Gamma_1)$. In fact, if $\| \partial_v z \|_{L^2(\Gamma_1)}^2 + \| z^2 \|_{L^2(\Gamma_1)}^2 = 0$, then $z = \partial_v z = 0$ on Γ_1. It follows that $\nabla_z = v\, \partial_v z = 0$ on Γ_1. Therefore, $z = 0$ in $H^1(\Gamma_1)$.

Next we introduce some operators (Ref.9) as follows:

(i) We set

$$Lz(s) = \int_0^\infty g'(s)z(s)\mathrm{d}s,$$
$$\mathcal{A}_0 = \Delta^2, \quad \mathcal{D}(\mathcal{A}_0) = \{w \in H^4(\Omega) \cap W | \mathcal{B}_1 w|_{\Gamma_1} = \mathcal{B}_2 w|_{\Gamma_1} = 0\}.$$

It is easy to know that $\mathcal{A}_0$ is a positive self-adjoint operator on $L^2(\Omega)$.
(ii) The Green operators N_1 and N_2 are introduced to describe the boundary conditions,

$$N_1 g = h \Leftrightarrow \begin{cases} \Delta^2 h = 0, & \text{in} \quad \Omega, \\ h = \partial_v h = 0, & \text{on} \quad \Gamma_0, \\ \mathcal{B}_1 h = g, & \text{on} \quad \Gamma_1, \\ \mathcal{B}_2 h = 0, & \text{on} \quad \Gamma_1, \end{cases}$$
$$N_2 g = h \Leftrightarrow \begin{cases} \Delta^2 h = 0, & \text{in} \quad \Omega, \\ h = \partial_v h = 0, & \text{on} \quad \Gamma_0, \\ \mathcal{B}_1 h = 0, & \text{on} \quad \Gamma_1, \\ \mathcal{B}_2 h = g, & \text{on} \quad \Gamma_1. \end{cases}$$

In terms of the regularity theory for the elliptic equations (Ref.6), we see that

$$N_1: L^2(\Gamma_1) \to H^{\frac{5}{2}}(\Omega) \quad \text{is continuous},$$
$$N_2: L^2(\Gamma_1) \to H^{\frac{7}{2}}(\Omega) \quad \text{is continuous}.$$

By these operators defined above, we may rewrite the system (1.1a-1.1f) as

$$y_{tt}(\cdot,t) + \mathcal{A}_0[y(\cdot,t) - N_1 L_z(\cdot,t,s) + N_2 L_z(\cdot,t,s) - N_2 u(\cdot,t,s)] = 0, \quad (2.1)$$

Where $z(\cdot,t,s) = y(x,t) - y(x,t-s), x \in \Gamma_1$. Considering $L^2(\Omega)$ as the pivot space: $[\mathcal{D}(\mathcal{A}_0)] \subset L^2(\Omega) \subset [\mathcal{D}(\mathcal{A}_0)]'$ and extending the $\mathcal{A}_0$ to be $\tilde{\mathcal{A}}_0: L^2(\Omega) \to [\mathcal{D}(\mathcal{A}_0)]'$, we can rewrite (4) as

$$y_{tt}(\cdot,t) = -\tilde{\mathcal{A}}_0 y(\cdot,t) + \tilde{\mathcal{A}}_0 N_1 L_z(\cdot,t) - \tilde{\mathcal{A}}_0 N_2 L_z(\cdot,t) + \tilde{\mathcal{A}}_0 N_2 u(\cdot,t) \in [\mathcal{D}(\mathcal{A}_0)]'. \quad (2.2)$$

Thus we can write the system (1.1a-1.1f) as a standard form of linear infinite-dimensional system in $\mathcal{H}$

$$\dot{Y}(t) = \mathcal{A}Y(t) + Bu \quad (2.3)$$

Where

$$Y(t) = \begin{bmatrix} y(\cdot,t) \\ y_t(\cdot,t) \\ z(\cdot,t,s) \end{bmatrix}, \quad z(\cdot,t,s) = y(x,t) - y(x,t-s),$$
$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\tilde{\mathcal{A}}_0 & 0 & \tilde{\mathcal{A}}_0 N_1 L - \tilde{\mathcal{A}}_0 N_2 L \\ 0 & I & -\dfrac{\partial}{\partial s} \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \{Y \in \mathcal{H} | \mathcal{A}Y \in \mathcal{H}\},$$

And

$$Bu = \begin{bmatrix} 0 \\ \tilde{\mathcal{A}}_0 N_2 u \\ 0 \end{bmatrix}, \quad B: L^2(\Gamma_1) \to [\mathcal{D}(\mathcal{A}^*)]' \text{is continuous}.$$

Finally, a direct computation gives

$$
\begin{aligned}
(N_2^*\mathcal{A}_0 f, g)_{L^2(\Gamma_1)} &= (\mathcal{A}_0 f, N_2 g)_{L^2(\Omega)} = (\Delta^2 f, N_2 g)_{L^2(\Omega)} \\
&= \int_\Omega f\overline{\Delta^2(N_2 g)}\mathrm{d}x - \int_{\Gamma_1}\left[f\overline{\mathcal{B}_2(N_2 g)} - \partial_v f\overline{\mathcal{B}_1(N_2 g)}\right]\mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_1}[\mathcal{B}_2 f\overline{(N_2 g)} - \mathcal{B}_1 f\,\overline{\partial_v(N_2 g)}]\mathrm{d}\Gamma \\
&= -\int_{\Gamma_1} f\overline{g}\mathrm{d}\Gamma,
\end{aligned}
$$

For all $f \in \mathcal{D}(\mathcal{A}_0)$ and $g \in L^2(\Gamma_1)$. Therefore, $N_2^*(\tilde{\mathcal{A}}_0)f = N_2^*\mathcal{A}_0 f = -f|_{\Gamma_1}, f \in \mathcal{D}(\mathcal{A}_0)$. It follows that

$$B^*\begin{bmatrix} w \\ v \\ z \end{bmatrix} = -v|_{\Gamma_1}, \quad \forall \begin{bmatrix} w \\ v \\ z \end{bmatrix} \in \mathcal{D}(\mathcal{A}^*). \qquad (2.4)$$

Now, let us consider a feedback control so that the input and output are collocated (Ref.7):

$$u = -kB^*(y, y_t, z)^T = ky_t|_{\Gamma_1}, \quad k \geq 0. \qquad (2.5)$$

The closed-loop system under this output feedback then becomes

$$y_{tt}(x,t) + \Delta^2 y(x,t) = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \qquad (2.6a)$$

$$y(x,t) = \partial_v y(x,t) = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \qquad (2.6b)$$

$$\mathcal{B}_1 y(x,t) - \int_0^\infty g'(s)\,\partial_v[y(x,t) - y(x,t-s)]\mathrm{d}s = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \qquad (2.6c)$$

$$\mathcal{B}_2 y(x,t) + \int_0^\infty g'(s)[y(x,t) - y(x,t-s)]\mathrm{d}s = ky_t(x,t), \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \qquad (2.6d)$$

$$y(x,0^+) = y_0(x), \quad y_t(x,0^+) = y_1(x), \qquad (2.6e)$$

$$y(x,-s) = \vartheta(x,t). \quad \text{for} \quad 0 < s < \infty, \qquad (2.6f)$$

The initial boundary problem (2.6) can be written as an evolutionary equation in $\mathcal{H}$:

$$\dot{Y}(t) = \mathcal{A}Y(t), \quad Y(0) = Y_0$$

Where $Y = (y, y_t, z), Y_0 = (y_0, y_1, y_0 - \vartheta)$ and

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & 0 \\ 0 & I & -\frac{\partial}{\partial s} \end{bmatrix}$$

With the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (w, v, z) \in \mathcal{H} \left| \begin{array}{l} \Delta^2 w \in L^2(\Omega), v \in W, z(\cdot) \in H^1(0,\infty; |g'(\cdot)|; H^1(\Gamma_1)), \\ z(0) = 0, [\mathcal{B}_1 w - \int_0^\infty g'(s)\,\partial_v z(s)\mathrm{d}s]_{\Gamma_1} = 0, \\ [\mathcal{B}_2 w + \int_0^\infty g'(s) z(s)\mathrm{d}s]_{\Gamma_1} = kv|_{\Gamma_1}, \end{array} \right. \right\}$$

Where

$$H^1(0,\infty; |g'(\cdot)|; H^1(\Gamma_1)) = \{z(s) \in Z | \frac{\partial}{\partial s} z(s) \in Z\}.$$

The following theorem ensures that the system (2.6) is well-posed in $\mathcal{H}$.

Theorem 2.1. Assume that the function g satisfies $(g1)$ through $(g3)$ and $k \geq 0$. Then the operator $\mathcal{A}$ generates a C_0-semigroup $S(t)$ of contraction on $\mathcal{H}$.

Proof. We first prove that $\mathcal{R}(I - \mathcal{A}) = \mathcal{H}$. Namely, we need to show that the following system of the equations

$w - v = f,$ (2.7a)
$v + \Delta^2 w = g,$ (2.7b)
$z(s) - v + \frac{\partial}{\partial s} z(s) = h(s)$ (2.7c)

has a solution $(u, v, z) \in \mathcal{D}(\mathcal{A})$ for every $(f, g, h) \in \mathcal{H}$. In fact, it follows from (2.6) that

$v = w - f \in W,$ (2.8a)
$w + \Delta^2 w = f + g \in L^2(\Omega),$ (2.8b)
$z(s) = (1 - e^{-s})w + (1 - e^{-s})f + \int_0^\infty e^{\tau - s} h(\tau) \mathrm{d}\tau \in Z.$ (2.8c)

Therefore, $v \in W$ and $z(\cdot) \in H^1(0, \infty; |g'(\cdot)|; H^1(\Gamma_1)), z(0) = 0.$ Furthermore, by (11b)-(11c) we have that for any $w \in W$ satisfying $\Delta^2 w \in L^2(\Omega)$ and $\mathcal{B}_1 w - \int_0^\infty g'(s)\, \partial_\nu z(s) \mathrm{d}s = 0, \mathcal{B}_2 w + \int_0^\infty g'(s) z(s) \mathrm{d}s = kv$, it has for all $\phi \in W$,

$$\begin{aligned} &\int_\Omega w\overline{\phi} \mathrm{d}x + a(w, \phi) + \int_{\Gamma_1} [(kw + Xw)\overline{\phi} + X\, \partial_\nu w\overline{\phi}] \mathrm{d}\Gamma \\ = \; &\int_\Omega (f + g)\overline{\phi} \mathrm{d}x + \int_{\Gamma_1} [(kf + Xf + \Psi)\, \overline{\partial_\nu \phi}] \mathrm{d}\Gamma \end{aligned} \qquad (2.9)$$

Where

$$X = -\int_0^\infty g'(s)(1 - e^{-s}) \mathrm{d}s \geq 0$$

And

$$\Psi = \int_0^\infty g'(s) \int_0^s e^{\tau - s} h(\tau) \mathrm{d}\tau \mathrm{d}s.$$

We see from the Lax-Milgram theorem (Ref.8) that the equation (2.9) admits a unique solution $w \in W.$ Combining this with $(2.8a)$ and (2.8c), we see that $(w, v, z) \in \mathcal{D}(\mathcal{A})$ solves the equation $(I - \mathcal{A})(w, v, z) = (f, g, h)$.

Next, for any $Y = (w, v, z) \in \mathcal{D}(\mathcal{A})$, it has

$$\begin{aligned} &\mathcal{R}e(\mathcal{A}Y, Y)_{\mathcal{H}} \\ = \; &-k \int_{\Gamma_1} |v|^2 d\Gamma - \frac{1}{2} \int_0^\infty \int_{\Gamma_1} g''(s)(|z(s)|^2 + |\, \partial_\nu z(s)|^2) \mathrm{d}\Gamma \mathrm{d}s \leq 0. \end{aligned} \qquad (2.10)$$

Hence $\mathcal{A}$ is dissipative. We see from the theorem 1.4.6 of Ref.8 that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Therefore, we can conclude by Lumer-Phillips theorem that $\mathcal{A}$ generates a C_o-semigroup of contractions on $\mathcal{H}$. The proof of Theorem 2.1 is complete now. □

3 A Variable Structural Control for the System

Let us establish a sliding model control for the system (??)

$$\begin{cases} \frac{\partial Y}{\partial t} = \mathcal{A}Y + Bw(Y, t) \\ Y(0) = Y_0 \end{cases} \qquad (3.1)$$

where B is a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}, w(Y, t)$ is the control of the system (3.1) that is not continuous on the manifold $S = CY = 0$, and C is a bounded linear operator with $S = S(Y) = CY \in R^n$.

Now, we consider the δ-neighborhood of sliding mode $S = CY = 0$, where $\delta > 0$ is an arbitrary given positive number. Using a continuous control $\widetilde{w}(z, t)$ to replace $w(z, t)$ in the system 3.1 yields

$$\begin{cases} \dot{Y} = \mathcal{A}Y + B\widetilde{w}(Y, t) \\ Y(0) = Y_0 \end{cases} \qquad (3.2)$$

where$\dot{Y} = \partial Y / \partial t$, and the solution of (3.2) belongs to the boundary layer $\| S(Y) \| \leq \delta$

Let $\dot{S}(Y) = C\dot{Y} = 0$. Applying C to the first equation of (3.1) leads to the following the equivalent control:

$$w_{eq}(Y,t) = -(CB)^{-1}C(\mathcal{A}Y)$$

With assumption that $(CB)^{-1}$ exists. Substitute $w_{eq}(Y,t)$ into 3.1 to find

$$\dot{Y} = [I - B(CB)^{-1}C]\mathcal{A}Y. \qquad (3.3)$$

Denote $P = B(CB)^{-1}C$ and $\mathcal{A}_0 = (I-P)\mathcal{A}$, then 3.1 becomes

$$\dot{Y} = \mathcal{A}_0 Y, \quad Y(0) = Y_0 \quad (3.4)$$

In the rest part of this paper, we are going to show that the actual sliding mode $Z(Y)$ will approach uniformly to the ideal sliding mode $\overline{Z}(Y)$ under certain conditions.

Lemma 3.1 If $(CB)^{-1}$ is a compact operator and $P\mathcal{A} = \mathcal{A}P$, then $\mathcal{A}_0 = (I-P)\mathcal{A}$ generates a C_0-semigroup $T_2(t)$ in $\mathcal{H}$ and $T_2(t) = (I-p)T_1(t)$, where $T_1(t)$ is the C_0-semigroup generated by $\mathcal{A}$.

Proof. Since $(CB)^{-1}$ is a compact operator, B and C are bounded linear operators, we see from the definition of P that P is compact, and therefor the range of $I-P$ is a closed subspace of $\mathcal{H}$. Since $P^2 = P$ and $(1-P)^2 = I-P$, $I-P$ can be viewed as the identity operator on $(I-P)\mathcal{H}$. It can be easily seen that $T_2(t) = (I-P)T_1(t)$ is a C_0-semigroup in $(I-P)\mathcal{H}$.

Next, we shall prove that the infinitesimal generator of $T_2(t)$ is $(I-P)\mathcal{A}$ and $\mathcal{D}((I-P)\mathcal{A}) = (I-P)\mathcal{D}(\mathcal{A})$.
In fact, for every $x \in (I-P)\mathcal{D}(\mathcal{A})$, there is a $x_1 \in \mathcal{D}(\mathcal{A})$ such that $x = (I-P)x_1$. It should be noted that $T_1(t)$ and $I-P$ are commutative because $\mathcal{A}$ and P are commutative. We see that

$$\lim_{t\to0^+}\frac{T_2(t)x - x}{t} = \lim_{t\to0^+}\frac{(I-P)T_1(t)(I-P)x_1 - (I-P)x_1}{t}$$
$$= \lim_{t\to0^+}\frac{(I-P)^2T_1(t)x_1 - (I-P)x_1}{t}$$
$$= \lim_{t\to0^+}\frac{(I-P)T_1(t)x_1 - (I-P)x_1}{t}$$
$$= (I-P)\lim_{t\to0^+}\frac{T_1(t)x_1 - x_1}{t}$$
$$= (I-P)\mathcal{A}x_1.$$

Let $\tilde{\mathcal{A}}$ be the infinitesimal generator of $T_2(t)$. Since the limit on the left exists, we can assert that $x \in \mathcal{D}(\tilde{\mathcal{A}})$ and $(I-P)\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\tilde{\mathcal{A}})$.

On the other hand, for any $x \in \mathcal{D}(\tilde{\mathcal{A}})$, since $\mathcal{D}(\tilde{\mathcal{A}}) \subseteq (I-P)\mathcal{H}$, there exists $\tilde{x} \in \mathcal{H}$, such that $x = (I-P)\tilde{x}$, and

$$\lim_{t\to0^+}\frac{T_2(t)x - x}{t} = \lim_{t\to0^+}\frac{T_2(t)(I-P)(\tilde{x}) - (I-P)(\tilde{x})}{t}$$
$$= \lim_{t\to0^+}\frac{(I-P)T_1(t)\tilde{x} - (I-P)\tilde{x}}{t}$$
$$= (I-P)\lim_{t\to0^+}\frac{T_1(t)\tilde{x} - \tilde{x}}{t}$$
$$= (I-P)\mathcal{A}\tilde{x}.$$

Since the limit of the left hand side exists, and so the limit of the right hand side exists, and $\tilde{x} \in \mathcal{D}(\mathcal{A})$ which implies that $\mathcal{D}(\tilde{\mathcal{A}}) \subseteq (I-P)\mathcal{D}(\mathcal{A})$. Thus, $\mathcal{D}(\tilde{\mathcal{A}}) = (I-P)\mathcal{D}(\mathcal{A})$and $\tilde{\mathcal{A}}$, the infinitesimal generator of $T_2(t)$, is $(I-P)\mathcal{A}$.
The proof of the lemma is complete.

Theorem 3.2 Suppose that in the system 3.1,
1. $(CB)^{-1}$ exists and it is compact,
2. $P\mathcal{A} = \mathcal{A}P$, where $P = B(CB)^{-1}C$.

Then for any solution $Y(t)$ of the system 3.4 satisfying$S(\overline{Y}_0) = 0, \overline{Y}_0 \in \mathcal{D}(\mathcal{A}_0)$ and $\| Y_0 - \overline{Y}_0 \| \leq \delta, Y_0 \in \mathcal{D}(\mathcal{A})$, we have

$$\lim_{\delta \to 0} \| z(t) - \overline{z}(t) \| = 0$$

Uniformly on $[0, T]$ for any positive number T.

Proof. We see from the Theorem 2.1 and Lemma 3.1 that $\mathcal{A}$ and $\mathcal{A}_0 = (I - P)\mathcal{A}$ are infinitesimal generators of C_0-semigroups $T_1(t)$ and $T_2(t)$ respectively. It follows from theory of semi group of linear operators that there are positive constants M_1, M_2, ω_1 and ω_2 such that

$$\| T_1(t) \| \leq M_1 e^{\omega_1 t}, \quad \| T_2(t) \| \leq M_2 e^{\omega_2 t}. \quad (0 \leq t \leq T) \qquad (3.5)$$

In the boundary layer $\| T_1(t) \| \leq \delta$, the equivalent control is

$$w_{eq}(Y, t) = -(CB)^{-1} C\mathcal{A}Y + (CB)^{-1} C\dot{Y} \qquad (3.6)$$

Substitute (3.6) into (3.1) to find

$$\dot{Y} = (I - P)\mathcal{A}Y + P\dot{Y} \quad (3.7)$$

Hence, the solution of (3.7) can be expressed as follows:

$$Y(t) = T_2(t)Y_0 + \int_0^t T_2(t - s)P\dot{Y}(s)\mathrm{d}s, \qquad (3.8)$$

And the solution of (3.4) can be written as

$$\overline{Y}(t) = T_2(t)\overline{Y}_0 \qquad (3.9)$$

Substracting (3.9) from (3.8) yields

$$Y(t) - \overline{Y}(t) = T_2(t)(Y_0 - \overline{Y}_0) + \int_0^t T_2(t - s)P\dot{Y}(s)ds \; (3.10)$$

Since$P\mathcal{A} = \mathcal{A}P$, we see that $PT_1(t) = PT_1(t)$. It should be emphasized that $(I - P)P = 0$ and $T_2(t) = (I - P)T_1(t)$, and consequently,

$$\int_0^t T_2(t - s)P\dot{Y}(s)ds = \int_0^t (I - P)T_1(t - s)P\dot{Y}(s)\mathrm{d}s$$
$$= \int_0^t T_1(t - s)(I - P)P\dot{Y}(s)\mathrm{d}s$$
$$= 0$$

It can be obtained from (3.10) and (3.5) that

$$\| Y(t) - \overline{Y}(t) \| \leq \| T_2(t) \| \| Y_0 - \overline{Y}_0 \| \leq M_2 e^{\omega_2 T} \| Y_0 - \overline{Y} \|,$$

Since$\| Y_0 - Y_0 \| \leq \delta$, we have

$$\| Y(t) - \overline{Y}(t) \| \leq M_2 e^{\omega_2 T} \delta.$$

Thus,

$$\lim_{\delta \to 0} \| Y(t) - \overline{Y}_0 \| = 0.$$

The proof of the theorem is complete.

We see from the Theorem 3.2 that the actual sliding mode can be approximated by ideal sliding mode in any accuracy.

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