Normal CR-Submanifolds of a Quasi Kaehlerian Manifold

Yong Wan¹ & Weizhi Chen²

Abstract:

In this paper, we establish a mathematical identity, which makes it possible to use the Gauss formula and Weingarten formula in the anti invariant distribution. Then we give some sufficient and necessary conditions for normal CR-submanifold of a quasi Kaehlerian manifold by both tensor $S$ and $S^*$ of type $(1, 2)$.

Keywords: quasi Kaehlerian manifold, CR-submanifold, normal, connection

1 Introduction

In this paper, all manifolds and morphisms are supposed to be differentiable of class $C^\infty$. Let $\overline{M}$ be a real $n$-dimensional connected differentiable manifold. $T(\overline{M})$ and $F(\overline{M})$ are respectively the tangle bundle to $\overline{M}$ and the algebra of differentiable functions on $\overline{M}$. Also, we denote by $\Gamma(H)$ the module of differentiable sections of a vector bundle $H$.

A linear connection on $\overline{M}$ is a mapping

$$\overline{\nabla}: \Gamma(T\overline{M}) \times \Gamma(T\overline{M}) \rightarrow \Gamma(T\overline{M}); \ (X, \ Y) \rightarrow \overline{\nabla}_X Y$$

satisfying the following conditions

$$(1) \ \overline{\nabla}_{f(X)+Y}(Z) = f \overline{\nabla}_X Z + \overline{\nabla}_Y Z,$$

¹ School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan, P. R. China. E-mail: wanyong870901@foxmail.com.
² School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan, P. R. China. E-mail: kalkuy@163.com.
\[ (2) \quad \nabla_x (fY + Z) = f\nabla_x Y + (Xf)Y + \nabla_x Z, \quad \text{for any } f \in F(\overline{M}) \text{ and } X, \ Y, \ Z \in \Gamma(T\overline{M}). \] 

The operator \( \nabla_x \) is called the covariant differentiation with respect to \( X \). Thus for any tensor field \( \Theta \) of type \((0, s)\) or \((1, s)\) we define the covariant differentiation \( \nabla_x \Theta \) of \( \Theta \) with respect to \( X \) by

\[ (\nabla_x \Theta)(X_1, X_2, \ldots, X_s) = \nabla_x (\Theta(X_1, X_2, \ldots, X_s)) - \sum_{i=1}^s \Theta(X_i, \nabla_x X_i, \ldots, X_s), \quad (1.1) \]

for any \( X_i \in \Gamma(T\overline{M}), i = 1, 2, \ldots, s \). A linear connection \( \nabla \) on \( \overline{M} \) is said to be a Riemannian connection if a Riemannian metric \( g \) satisfying

\[ Xg(Y, Z) = g(\nabla_x Y, Z) + g(Y, \nabla_x Z), \quad (1.2) \]

for any \( X, \ Y \in \Gamma(T\overline{M}) \). An almost complex structure on \( \overline{M} \) is a tensor field \( J \) of type \((1, 1)\) on \( \overline{M} \) such that at every point \( x \in \overline{M} \) we have \( J^2 = -I \), where \( I \) denotes the identity transformation of \( T_x\overline{M} \). A manifold \( \overline{M} \) endowed with an almost complex structure is called an almost complex manifold. The covariant derivative of \( J \) is defined by

\[ (\nabla_x J)Y = \nabla_x JY - J\nabla_x Y, \quad (1.3) \]

for any \( X, \ Y \in \Gamma(T\overline{M}) \). More, we define the torsion tensor of \( J \) or the Nijenhuis tensor of \( J \) by

\[ [J, J](X, Y) = [JX, JY] - [X, JY] - [JX, Y] - J[X, JY], \quad (1.4) \]

for any \( X, Y \in \Gamma(T\overline{M}) \), where \([X, Y]\) is the Lie bracket of vector fields \( X \) and \( Y \), that is,

\[ [X, Y] = \nabla_x Y - \nabla_y X. \]

A Hermitian metric on an almost complex manifold \( \overline{M} \) is a Riemannian metric \( g \) satisfying

\[ g(JX, JY) = g(X, Y), \quad (1.5) \]

for any \( X, Y \in \Gamma(T\overline{M}) \). An almost complex manifold endowed with a Hermitian metric is said to be an almost Hermitian manifold. Definition 1.1([3]). An almost Hermitian manifold \( \overline{M} \) with Levi-Civita connection \( \nabla \) is called a quasi Kählerian manifold if we have \( (\nabla_x J)Y + (\nabla_x J)JY = 0 \),

\[ (1.6) \]

for any \( X, Y \in \Gamma(T\overline{M}) \). Definition 1.2([1]). An almost Hermitian manifold \( \overline{M} \) with Levi-Civita connection \( \nabla \) is called a Kählerian manifold if we have \( \nabla_x J = 0 \),

\[ (1.7) \]

for any \( X \in \Gamma(T\overline{M}) \). Obviously, a Kählerian manifold is a quasi Kählerian manifold. Let \( M \) be an \( m \)-dimensional Riemannian submanifold of an \( n \)-dimensional Riemannian manifold \( \overline{M} \). We denote by \( TM^\perp \) the normal bundle to \( M \) and by \( g \) both metric on \( M \) and \( \overline{M} \). Also, we denote by \( \nabla \) the Levi-Civita connection on \( \overline{M} \), denote by \( \nabla \) the induced connection on \( M \), and denote by \( \nabla^\perp \) the induced normal connection on \( M \).

Then, for any \( X, Y \in \Gamma(TM) \) we have

\[ \nabla_x Y = \nabla_x Y + h(X, Y), \quad (1.8) \]
where $h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.8) is called the Gauss formula and $h$ is called the second fundamental form of $M$. Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$ we denote by $-A_v X$ and $\nabla^\perp_x V$ the tangent part and normal part of $\nabla_x V$ respectively. Then we have $\nabla_x V = -A_v X + \nabla^\perp_x V$. (1.9)

Thus, for any $V \in \Gamma(TM^\perp)$ we have a linear operator, satisfying

$$g(A_v X, Y) = g(X, A_v Y) = g(h(X, Y), V).$$

The equation (1.9) is called the Weingarten formula. An $m$-dimensional distribution on a manifold $\overline{M}$ is a mapping $D$ defined on $\overline{M}$, which assigns to each point $x$ of $\overline{M}$ an $m$-dimensional linear subspace $D_x$ of $T_x \overline{M}$. A vector field $X$ on $\overline{M}$ belongs to $D$ if we have $X_x \in D_x$ for each $x \in \overline{M}$. When this happens we write $X \in \Gamma(D)$. The distribution $D$ is said to be differentiable if for any $x \in \overline{M}$ there exist $m$ differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of $x$. From now on, all distributions are supposed to be differentiable of class $C^\infty$. Definition 1.3([1]). Let $\overline{M}$ be a real $n$-dimensional almost Hermitian manifold with almost complex structure $J$ and with Hermitian metric $g$. Let $M$ be a real $m$-dimensional Riemannian manifold isometrically immersed in $\overline{M}$. Then $M$ is called a CR-submanifold of $\overline{M}$ if there exist a differentiable distribution $D : x \to D_x \subset T_x M$, on $M$ satisfying the following conditions: (1) $D$ is holomorphic, that is, $J(D_x) = D_x$, for each $x \in M$,

(2) the complementary orthogonal distribution $D^\perp : x \to D^\perp_x \subset T_x M$, is anti-invariant, that is, $J(D^\perp_x) \subset T_x M^\perp$, for each $x \in M$. Now let $M$ be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold $\overline{M}$. For each vector field $X$ tangent to $M$, we put $JX = \phi X + \omega X$, (1.11)

where $\phi X$ and $\omega X$ are respectively the tangent part and the normal part of $JX$. We denote by $P$ and $Q$ respectively the projection morphisms of $TM$ to $D$ and $D^\perp$, that is,

$$X = PX + QX,$$

(1.12)

for any $X \in \Gamma(TM)$. Then we have

$$\phi X = JPX$$

(1.13)

and

$$\omega X = JQX,$$

(1.14)

for any $X \in \Gamma(TM)$. Moreover, we have

$$\phi^2 = -P$$

(1.15)

and
\[ \phi^3 + \phi = 0. \quad (1.16) \]

Next, for each vector field \( V \) normal to \( M \), we put
\[ JV = BV + CV, \quad (1.17) \]
where \( BV \) and \( CV \) are respectively the tangent part and the normal part of \( JV \).

We take account of the decomposition \( T\overline{M} = D \oplus D^\perp \oplus JD^\perp \oplus \nu \). Obviously, we have \( \phi X \in \Gamma(D), \ \omega X \in \Gamma(JD^\perp), \ BV \in \Gamma(D^\perp) \) and \( CV \in \Gamma(\nu) \), for any \( X \in \Gamma(TM) \) and \( V \in \Gamma(JD^\perp \oplus \nu) \). Further, we obtain \( B \circ \omega = -Q \).

The covariant derivative of \( \phi \) is defined by
\[ (\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \quad (1.18) \]
for any \( X, \ Y \in \Gamma(TM) \). On the other hand, the covariant derivative of \( \omega \) is defined by
\[ (\nabla_X \omega)Y = \nabla^\perp_X \omega Y - \omega \nabla_X Y, \quad (1.19) \]
for any \( X, \ Y \in \Gamma(TM) \). The exterior derivative of \( \omega \) is given by
\[ d\omega(X, \ Y) = \frac{1}{2} \{ \nabla^\perp_X \omega Y - \nabla^\perp_Y \omega X - \omega([X, \ Y]) \}, \quad (1.20) \]
for any \( X, \ Y \in \Gamma(TM) \).

Remark: The more details of exterior derivative is found in [2]. The Nijenhuis tensor of \( \phi \) is defined by
\[ [\phi, \ \phi](X, \ Y) = [\phi X, \ \phi Y] + \phi^2[X, \ Y] - \phi[\phi X, \ Y] - \phi[X, \ \phi Y], \quad (1.21) \]
for any \( X, \ Y \in \Gamma(TM) \), where \([X, \ Y]\) is the Lie bracket of vector fields \( X \) and \( Y \). We define two the tensor fields \( S \) and \( S^* \) respectively by
\[ S(X, \ Y) = [\phi, \ \phi](X, \ Y) - 2Bd\omega(X, \ Y), \quad (1.22) \]
and
\[ S^*(Y, \ X) = (L_\phi)X = [Y, \ \phi X] - \phi[Y, \ X], \quad (1.23) \]
for any \( X, \ Y \in \Gamma(TM) \). Definition 1.4([1]). The CR-submanifold \( M \) is said to be normal if
\[ S(X, \ Y) = 0, \quad (1.24) \]
for any \( X, \ Y \in \Gamma(TM) \). Definition 1.5. The CR-submanifold \( M \) is said to be mixed normal if
\[ S(X, \ Y) = 0, \quad (1.25) \]
for any \( X \in \Gamma(D), \ Y \in \Gamma(D^\perp) \).

2 Main Results

**Lemma 2.1.** Let \( \overline{M} \) be a quasi Kaehlerian manifold. Then we have
\[(\nabla_X J)Y - (\nabla_Y J)X = \frac{1}{2} J[J, J](X, Y), \quad (2.1)\]

for any \(X, Y \in \Gamma(TM)\).

Proof: For any \(X, Y \in \Gamma(TM)\). From (1.4) and (1.3) we acquire
\[ [J, J](X, Y) = (\nabla_X J)Y - (\nabla_Y J)X + J(\nabla_Y J)X - J(\nabla_X J)Y. \quad (2.2) \]

Using (2.2), (1.6) and (1.3) we have
\[ [J, J](X, Y) = (\nabla_X J)JY - (\nabla_Y J)JX + J(\nabla_Y J)X - J(\nabla_X J)Y \]
\[ = 2J((\nabla_Y J)X - (\nabla_X J)Y). \quad (2.3) \]

(2.3) follows that (2.1) holds. \(Q.E.D.\)

Lemma 2.2. Let \(M\) be a quasi Kaehlerian manifold. Then we have
\[ (\nabla_X J)Y - (\nabla_Y J)X = \frac{1}{2} [J, J](X, Y), \quad (2.4) \]

for any \(X, Y \in \Gamma(TM)\). Proof: For any \(X, Y \in \Gamma(TM)\). From (1.6) we get
\[ (\nabla_X J)Y - (\nabla_Y J)X = -(\nabla_X J)J^2Y + (\nabla_Y J)J^2X \]
\[ = (\nabla_X J)JY - (\nabla_Y J)JX. \quad (2.5) \]

Using (1.3) in (2.5) we obtain
\[ (\nabla_X J)Y - (\nabla_Y J)X = -J((\nabla_X J)Y - (\nabla_Y J)X). \quad (2.6) \]

(2.4) comes from (2.6). \(Q.E.D.\)

Lemma 2.3. Let \(M\) be a quasi Kaehlerian manifold. Then we have
\[ (\nabla_X \phi)Y = A_{\phi X}X + Bh(X, Y) + \nabla_{\phi X}Y + \phi \nabla_{\phi X}Y \]
\[ + Bh(\phi X, \phi Y) - \phi A_{\phi X} \phi X + B\nabla_{\phi X} \phi Y, \quad (2.7) \]

\[ (\nabla_X \omega)Y = -h(X, \phi Y) + Ch(X, Y) + h(\phi X, Y) + \omega \nabla_{\phi X} \phi Y \]
\[ + Ch(\phi X, \phi Y) - \omega A_{\phi X} \phi X + C\nabla_{\phi X} \phi Y, \quad (2.8) \]

for any \(X \in \Gamma(D), Y \in \Gamma(TM)\).

Proof: For any \(X \in \Gamma(D), Y \in \Gamma(TM)\). Using (1.6) and (1.3), we have
\[ (\nabla_X JY - J\nabla_X Y) + (\nabla_{\phi X} Y - J\nabla_{\phi X} JY) = 0. \quad (2.9) \]

Taking into account (1.11), (2.9) becomes
\[ (\nabla_X \phi Y + \nabla_X \omega Y) - J\nabla_X Y - \nabla_{\phi X} Y - J(\nabla_{\phi X} \phi Y + \nabla_{\phi X} \omega Y) = 0. \quad (2.10) \]

Taking account of (1.8) and (1.9), (2.10) changes into
\[ \nabla_x \phi Y + h(X, \phi Y) - A_{aoY} X + \nabla_x^+ \omega Y - J \nabla_x Y - J h(X, Y) - \nabla_{\phi \omega} Y - h(\phi X, Y) \]
\[ - J \nabla_{\phi \omega} Y - h(\phi X, Y) + J A_{aoY} \phi X - J \nabla_{\phi \omega} \omega Y = 0. \]  
(2.11)

According to (1.11) and (1.17), (2.11) turns into
\[ \nabla_x \phi Y + h(X, \phi Y) - A_{aoY} X + \nabla_x^+ \omega Y - \phi \nabla_x Y - \omega \nabla_x Y - B h(X, Y) - Ch(X, Y) \]
\[ - \nabla_{\phi \omega} Y - h(\phi X, Y) - \phi \nabla_{\phi \omega} \phi Y - \omega \nabla_{\phi \omega} \phi Y - B h(\phi X, \phi Y) - Ch(\phi X, \phi Y) \]
\[ + \phi A_{aoY} \phi X + \omega A_{aoY} \phi X - B \nabla_{\phi \omega}^+ \omega Y - C \nabla_{\phi \omega}^+ \omega Y = 0. \]  
(2.12)

By comparing to the tangent part and the normal part in (2.12), we get
\[ \nabla_x \phi Y - A_{aoY} X - \phi \nabla_x Y - B h(X, Y) - \nabla_{\phi \omega} Y - \phi \nabla_{\phi \omega} \phi Y - B h(\phi X, \phi Y) \]
\[ + \phi A_{aoY} \phi X - B \nabla_{\phi \omega}^+ \omega Y = 0 \]  
(2.13)

And
\[ h(X, \phi Y) + \nabla_x^+ \omega Y - \omega \nabla_x Y - Ch(X, Y) - h(\phi X, Y) - \omega \nabla_{\phi \omega} \phi Y - Ch(\phi X, \phi Y) \]
\[ + \omega A_{aoY} \phi X - C \nabla_{\phi \omega}^+ \omega Y = 0. \]  
(2.14)

By (2.13) and (1.18) we have (2.7). Also, by (2.14) and (1.19) we get (2.8). Q.E.D.

Lemma 2.4([1]). Let \( M \) be a CR-submanifold of an almost Hermitian manifold \( \overline{M} \). Then we have
\[ S(X, Y) = (\nabla_{\phi \omega} \phi) Y - (\nabla_{\phi \omega} \phi) X + \phi \{ (\nabla_{\phi \omega} \phi) X - (\nabla_{\phi \omega} \phi) Y \} - B \{ (\nabla_{\phi \omega} \phi) Y - (\nabla_{\phi \omega} \phi) X \}, \]  
(2.15)

for any \( X, Y \in \Gamma(TM) \).

Lemma 2.5. Let \( M \) be a CR-submanifold of a quasi Kählerian manifold \( \overline{M} \). Then we have
\[ S(X, Y) = A_{aoY} \phi X - \phi A_{aoY} X - A_{aoX} \phi Y + \phi A_{aoX} Y + (\nabla_{\phi \omega} J) Y - (\nabla_{\phi \omega} J) X \]
\[ = -\frac{1}{2} \phi (J [J, J] (X, Y))^{\top} - \frac{1}{2} B(J [J, J] (X, Y))^{\top}, \]  
(2.16)

for any \( X, Y \in \Gamma(TM) \).

Proof: For any \( X, Y \in \Gamma(TM) \). Taking into account (1.3), (1.11), (1.8), (1.9) and (1.17), we have
\[ (\nabla_{\phi \omega} J) Y = \nabla_{\phi \omega} \phi Y + \omega Y - J (\nabla_{\phi \omega} Y + h(X, Y)) \]
\[ = \nabla_{\phi \omega} \phi Y + h(X, \phi Y) - A_{aoY} X + \nabla_{\phi \omega} \omega Y \]
\[ - \phi \nabla_{\omega Y} Y - \omega \nabla_{\phi \omega} Y - B h(X, Y) - Ch(X, Y). \]  
(2.17)

By comparing to the tangent part and the normal part in (2.17), we obtain
\[ ((\nabla_{\phi \omega} J) Y) = \nabla_{\phi \omega} \phi Y - A_{aoY} X - \phi \nabla_{\phi \omega} Y - B h(X, Y) \]  
(2.18)

and
\[(\overline{\nabla}_X J)Y^\perp = h(X, \phi Y) + \nabla^\perp_X \omega Y - \omega \overline{\nabla}_X Y - Ch(X, Y). \quad (2.19)\]

Combining (1.18) and (2.18), we have
\[(\nabla_X \phi)Y = A_{\omega \phi} X + Bh(X, Y) + (\overline{\nabla}_X J)Y^T. \quad (2.20)\]

Combining (1.19) and (2.19), we get
\[(\nabla_X \omega)Y = -h(X, \phi Y) + Ch(X, Y) + (\overline{\nabla}_X J)Y^\perp. \quad (2.21)\]

Taking account of (2.20) and (2.21), (2.15) becomes
\[S(X, Y) = A_{\omega \phi} \phi X + ((\overline{\nabla}_x J)Y)^T - A_{\omega \alpha} \phi Y - ((\overline{\nabla}_{\phi \gamma} J)X)^T + \phi A_{\omega \alpha} Y + \phi ((\overline{\nabla}_Y J)X)^T - \phi A_{\omega \alpha} X - \phi ((\overline{\nabla}_X J)Y)^T - B((\overline{\nabla}_X J)Y)^\perp + B((\overline{\nabla}_Y J)X)^\perp. \quad (2.22)\]

Combining (2.22) and (2.1), we obtain our conclusion (2.16).

**Theorem 2.2.** Let \(M\) be a CR-submanifold of a quasi Kaehlerian manifold \(\overline{M}\). Then \(M\) is normal if and only if we have
\[0 = A_{\omega \phi} \phi X - \phi A_{\omega \phi} X - A_{\omega \alpha} \phi Y + \phi A_{\omega \alpha} Y + ((\overline{\nabla}_x J)Y - (\overline{\nabla}_{\phi \gamma} J)X)^T - \frac{1}{2} \phi (J[J, J](X, Y))^T - \frac{1}{2} B(J[J, J](X, Y))^\perp, \quad (2.23)\]

for any \(X, Y \in \Gamma(TM)\).

**Proof:** Taking account of Definition 1.4 and Lemma 2.5, our conclusion holds. \(\text{Q.E.D.}\)

**Corollary 2.1.** Let \(M\) be a CR-submanifold of a Kaehlerian manifold \(\overline{M}\). Then \(M\) is normal if and only if we have
\[A_{\omega \phi} \phi X - \phi A_{\omega \phi} X - A_{\omega \alpha} \phi Y + \phi A_{\omega \alpha} Y = 0, \quad (2.24)\]

for any \(X, Y \in \Gamma(TM)\).

**Proof:** Since a Kaehlerian manifold \(\overline{M}\) satisfies
\[\overline{\nabla}_X J = 0, \quad [J, J](X, Y) = 0, \]
for any \(X, Y \in \Gamma(TM)\), taking account of Theorem 2.1, Corollary 2.1 holds. \(\text{Q.E.D.}\)

**Corollary 2.2 (Bejancu[1]).** Let \(M\) be a CR-submanifold of a Kaehlerian manifold \(\overline{M}\). Then \(M\) is normal if and only if we have
\[A_{\omega \phi} \phi X = \phi A_{\omega \phi} X, \quad (2.25)\]

for any \(X \in \Gamma(D), Y \in \Gamma(D^\perp)\). Theorem 2.2. Let \(M\) be a CR-submanifold of a quasi Kaehlerian manifold \(\overline{M}\) and
\[A_{\omega \alpha} X + \overline{\nabla}_{\phi \alpha} Y \in \Gamma(D^\perp), \quad (2.27)\]

for any \(X, Y \in \Gamma(TM)\). Then \(M\) is normal if and only if we have
\[h(X, Y) \in \Gamma(v), \quad (2.28)\]
for any $X \in \Gamma(D), \; Y \in \Gamma(D^\perp)$.

Proof: For any $X \in \Gamma(D), \; Y \in \Gamma(D^\perp)$. By using (2.26) in (2.16) we obtain

$$S(X, \; Y) = A_{aoY} \phi X - \phi A_{oY} X + ((\overline{\nabla}_{\phi Y} J)Y)^T. \quad (2.29)$$

Taking into account (1.8), (1.9), (1.11) and (1.17), (1.3) becomes

$$(\overline{\nabla}_{\phi Y} J)Y = -A_{aoY} \phi X + \nabla_{\phi Y} Y - \phi \nabla_{\phi Y} Y - \omega \overline{\nabla}_{\phi Y} Y - Bh(\phi X, \; Y) - Ch(\phi X, \; Y). \quad (2.30)$$

By comparing to the tangent part and the normal part in (2.30), we get

$$((\overline{\nabla}_{\phi Y} J)Y)^T = -A_{aoY} \phi X - \phi \nabla_{\phi Y} Y - Bh(\phi X, \; Y). \quad (2.31)$$

From (2.29) and (2.31), we obtain

$$S(X, \; Y) = -A_{aoY} \phi X - \phi \nabla_{\phi Y} Y - Bh(\phi X, \; Y). \quad (2.32)$$

Suppose $M$ is normal CR-submanifold of $\overline{M}$. For any $X \in \Gamma(D), \; Y \in \Gamma(D^\perp)$, then from (2.32) and Definition 1.4 we have $\phi(A_{aoY} X + \nabla_{\phi Y} Y) = 0$ \hspace{1cm} (2.33)

And $Bh(\phi X, \; Y) = 0$. \hspace{1cm} (2.34)

From (2.33) we obtain (2.27), correspondingly, from (2.34) we get (2.28). Conversely, if (2.27) and (2.28) are satisfied.

Now, we shall prove $S = 0$ by means of the decomposition $TM = D \oplus D^\perp$. First, for any $X \in \Gamma(D), \; Y \in \Gamma(D^\perp)$, from (2.27) we obtain (2.33), correspondingly, from (2.28) we get (2.34). Taking account of (2.33) and (2.34), (2.32) becomes $S(X, \; Y) = 0, \; \forall X \in \Gamma(D), \; Y \in \Gamma(D^\perp)$. Next, for any $X, \; Y \in \Gamma(D)$, by using (2.26), (2.16) changes into $S(X, \; Y) = ((\overline{\nabla}_{\phi Y} J)Y)^T - ((\overline{\nabla}_{\phi Y} J)X)^T$

$$= ((\overline{\nabla}_{JX} Y - (\overline{\nabla}_{JY} J)X)^T. \quad (2.35)$$

From (2.4) and (2.26), (2.35) becomes $S(X, \; Y) = 0, \; \forall X, \; Y \in \Gamma(D)$. Finally, for any $X, \; Y \in \Gamma(D^\perp)$, in accordance with (2.26), (2.16) changes over $S(X, \; Y) = -A_{aoY} X + \phi A_{oY} Y \in \Gamma(D)$.

$$\forall Z \in \Gamma(D), \; \text{on the basis of (2.36), (1.11) and (1.10), we have}$$

$$g(S(X, \; Y), \; Z) = g(-\phi A_{oY} X, \; Z) + g(\phi A_{oY} Y, \; Z)$$

$$= g(h(X, \; \phi Z), \; \omega Y) - g(h(Y, \; \phi Z), \; \omega Y). \quad (2.37)$$

Using (2.28) in (2.37), we get

$$g(S(X, \; Y), \; Z) = 0. \quad (2.38)$$

That is, $S(X, \; Y) = 0, \; \forall X, \; Y \in \Gamma(D)$. 


From the above three conclusions we know $S(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$. Thus, the CR-submanifold $M$ is normal. Q.E.D. Theorem 2.3. Let $M$ be a CR-submanifold of a quasi Kaehlerian manifold $\overline{M}$ with following conditions satisfying $\nabla_X Y \in \Gamma(D)$ for any $X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$. Then $M$ is mixed normal if and only if we have
$$S^*(Y, X) = 0,$$ (2.41)
for any $X \in \Gamma(D), \ Y \in \Gamma(D^\perp)$.

Proof: For any $X \in \Gamma(D), \ Y \in \Gamma(D^\perp)$. According to (1.18), (2.15) becomes
$$S(X, Y) = \phi(\phi[X, Y] - [\phi X, Y]) - B(\nabla_X \omega)Y + B(\nabla_Y \omega)X. \quad (2.42)$$

Taking into account (1.19), (2.8) and $B \circ C = 0$, (2.42) changes into
$$S(X, Y) = \phi(\phi[X, Y] - [\phi X, Y]) - Bh(\phi X, Y) + B\alpha\alpha_{\phi} \phi X - B\alpha V, X. \quad (2.43)$$

Taking account of (1.23), (2.40) and $B \circ \omega = -Q$, (2.43) changes over
$$S(X, Y) = \phi S^*(Y, X) - QA_{\omega} \phi X + QV, Y X. \quad (2.44)$$
\[ \forall U \in \Gamma(D^\perp), \text{ combining (1.12), (1.10) and (2.40), we have} \]
$$g(QA_{\omega} \phi X, U) = g(A_{\omega} \phi X, U) = g(h(\phi X, U), \omega Y) = 0. \quad (2.45)$$

(2.45) leads to $QA_{\omega} \phi X = 0, \ \forall X \in \Gamma(D), \ Y \in \Gamma(D^\perp). \quad (2.46)$

Combining (2.44) and (2.46), we get $S(X, Y) = \phi S^*(Y, X) + QV, Y X, \ \forall X \in \Gamma(D), \ Y \in \Gamma(D^\perp). \quad (2.47)$

Suppose $M$ is mixed normal CR-submanifold of $\overline{M}$. For any $X \in \Gamma(D), \ Y \in \Gamma(D^\perp)$, then from (2.47) it follows
$$\phi S^*(Y, X) = 0 \quad (2.48)$$

and
$$QV, Y X = 0. \quad (2.49)$$

Based on (2.48) we obtain
$$S^*(Y, X) \in \Gamma(D^\perp), \quad (2.50)$$
for any $X \in \Gamma(D), \ Y \in \Gamma(D^\perp)$. On the other hand, taking into account (2.39) and (2.49), (1.23) becomes
$$S^*(Y, X) = \nabla_Y \phi X - \nabla_{gX} Y - \phi[Y, X] \in \Gamma(D), \quad (2.51)$$
for any $X \in \Gamma(D), \ Y \in \Gamma(D^\perp)$. Taking account of (2.50) and (2.51), we get that (2.41) holds.

Conversely, if (2.41) is satisfied. For any $X \in \Gamma(D), \ Y \in \Gamma(D^\perp)$, combining (1.15) and (1.12), (1.23) changes into
\[ S^\top(Y, -\phi X) = [Y, X] - \phi[Y, X] = P[Y, X] - \phi[Y, X] + Q[Y, X]. \] (2.52)

By using (2.41) in (2.52), we have \[ Q[Y, X] = 0, \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp). \] (2.53)

From (2.53) and (2.39), we obtain \[ Q \nabla_Y X = 0, \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp). \] (2.54)

Combining (2.41) and (2.54), (2.47) becomes \[ S(Y, X) = 0, \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp). \] (2.55)

Relying on Definition 1.5, \( M \) is mixed normal. Q.E.D.

References

Kentaro Yano and Masahiro Kon, Structures on manifolds, World Scientific Publishing Company, Singapore (1984), pp. 16-17