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A Class of Piecewise Linear Maps

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Abstract

Piecewise linear functions defined by p-maps, linear only on a subset of r vectors and components, are introduced. Universal properties for this map are proved. Spaces of extensions of differential forms by piecewise linear functions are considered

Keywords: master's degree, doctoral degree, mathematical sciences, mathematics

Introduction

Piecewise linear functions are useful in several different contexts, piecewise linear manifolds, computer science or convex analysis are examples. A definition of a piecewise linear function is the following, see [8]. Let C a closed convex domain in $\Re^d$, a function $\Phi: C \to \Re$ is said to be piecewise linear if there is a finite family Q of closed domains such that $C = \cup Q$ and Φ is linear on every domain in Q. A linear function ϕ on $\Re^d$ which coincides with Φ on some $Q_i \in Q$ is said to be a component of Φ. In this paper is considered a more general class of piecewise linear functions. It is defined the set of maps $SW(E^m, T)$ which are linear only on a subset of r vectors and components.

Then an exponential function F is defined from linear spaces to the set $SW(E^m, T)$. It is proved the uniqueness and existence of a function * as universal element for the function F. It is defined a r-subset wise linear skew symmetric $\Phi = \sum_{\mu,\nu} \lambda_\nu^\mu \phi$ map and it is proved that this is completely determined by its values for λ_ν^μ and on a basis of E. A r-determinant function is defined as a r-subset wise linear skew symmetric map $\Phi: E^m \to \Gamma$, where Γ is an arbitrary field of characteristic 0. Some properties of r-determinant maps are considered. It is defined the adjoint for a linear map $\psi \in L(E, F)$, where E and F are linear spaces, and the development of a r-determinant function by r- cofactors. Extensions of differential forms are defined by r-subset wise skew symmetric maps. Basis and spaces of generalized differential forms are studied.

2. R-Subset wise Linear Mappings

Some properties of linear functions are extended to mappings which are linear only on subsets of r variables. Γ Denotes an arbitrarily chosen field such that $char\Gamma = 0$.

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The multindex I_r^n of lenght r is defined by

$$I_r^n = \{(i_1,\dots,i_r):\ 1 \le i_1 < i_2 < \cdots < i_r \le n\}$$

Besides, for a fixed natural k

$$(I_r^n)_k = \{(i_1,\dots,i_p,\dots,i_r): 1 \le i_1 < \cdots < i_p = k < \cdots \le i_r \le n, \text{ where } 1 \le k \le n\}$$

for the indices $j_1,\dots,j_k \in I_k^n$

$$(I_r^n)_{j_1,\dots,j_k} = \{(i_1,\dots,i_{p_1},\dots,i_{p_k},\dots,i_r): 1 \le i_1 < \cdots < i_{p_1} = j_1 < \dots < i_{p_k} = j_k < \cdots \le i_r \le n\}$$

Let $\{e_\nu\}$ be a basis of an n-dimensional vector space E and let $x^\mu = \sum_{\nu=1}^n x_\nu^\mu e_\nu$ be vectors of E, $n \ge 1$.

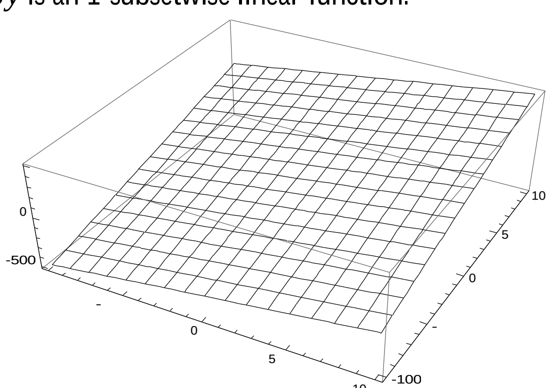
Definition 2.1*Let $L(E^r, T)$ be the space of linear mappings of E^r into the vector space T. Consider a mapping*

$$\begin{cases} \Phi: & E^m \to T \\ \Phi: & (x_1,\dots,x_m) \mapsto \sum_{\mu,\nu} \lambda_\nu^\mu \phi(x_\nu^{\mu_1} e_\nu,\dots,x_\nu^{\mu_r} e_\nu) \quad 1 \le r \le n, 1 \le r \le m,\ \lambda_\nu^\mu \in \Gamma \end{cases}$$

Where the sum is over every system of indices $\mu = \mu_1,\dots,\mu_r \in I_r^m$, $\nu = \nu_1,\dots,\nu_r \in I_r^n$. If $n = m$ then $r < n = m$. The sum $(x_{\nu_1}^{\mu_i} e_{\nu_1} + \cdots + x_{\nu_r}^{\mu_i} e_{\nu_r})$ is denoted in short by $x_\nu^{\mu_i} e_\nu$, and $\phi : E^r \to T$ is an r-linear mapping. Then Φ is said to be r-linear with respect to the r-subsets of vectors and components, that is, an r-subsetwise linear mapping. The linear mappings ϕ are the components of Φ.

Example 2.1 *The function $\Phi : \Re^{1\times 2} \to \Re$ defined by*

$\Phi(x, y) = 2x + 3y$ is an 1-subsetwise linear function.



Graph of the function Φ. (Obtained by Mathematica).

Example 2.2 *The map* $\Phi : (\Re^2)^3 \to \Re^{2\times 2}$ *defined by*

$$\Phi[(x_{11},x_{21}),(x_{12},x_{22}),(x_{13},x_{23})] = \lambda^{12}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \lambda^{13}\begin{pmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{pmatrix} + \lambda^{23}\begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix} \qquad \lambda^{\mu} \in \Re$$

is an 2-subsetwise linear map.

Example 2.3 *Let* $f_1,\dots,f_r$ *be a linearly independent set of the space* $L(E^r,T)$, *a r-subsetwise linear map is defined by*

$$\Phi(x_1,\dots,x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu (f_1(x_\nu^{\mu_1} e_\nu)\cdot f_2(x_\nu^{\mu_2} e_\nu)\cdots f_r(x_\nu^{\mu_r} e_\nu)) \qquad \lambda_\nu^\mu \in \Gamma$$

Theorem 2.1 *An r-subsetwise linear mapping* Φ, *with* $r < m$, *is not linear Proof.* For any r-subsetwise linear mapping Φ, $r < m$,

$$\begin{aligned}\Phi(x_1,\dots,x_i+y_i,\dots,x_m) &= \sum_{\mu,\nu}\lambda_\nu^\mu \phi(x_\nu^{\mu_1}e_\nu,\dots,x_\nu^i e_\nu,\dots,x_\nu^{\mu_r}e_\nu) + \sum_{\mu,\nu}\lambda_\nu^\mu \phi(x_\nu^{\mu_1}e_\nu,\dots,y_\nu^i e_\nu,\dots,x_\nu^{\mu_r}e_\nu) \\ &\neq \Phi(x_1,\dots,x_i,\dots,x_m) + \Phi(x_1,\dots,y_i,\dots,x_m)\end{aligned}$$

In the first sum on the right side $\mu = \mu_1,\dots,i,\dots,\mu_r \in I_r^m$. Unlike, in the second sum $\mu = \mu_1,\dots,i,\dots,\mu_r \in (I_r^m)_i$, so this sum cannot be $\Phi(x_1,\dots,y_i,\dots,x_m)$. □

As a special case, if $r = m$ then Φ is linear.

If $t : T \to H$ is linear and Φ is r-swlin (subsetwise linear) map, then

$$t\Phi = t(\sum \lambda_\nu^\mu \phi) = \sum \lambda_\nu^\mu t\phi$$

and $t\Phi$ is a r-swlin map.

By the set $SW(E^m,T)$ of the r-swlin maps, the following exponential functor F, from linear spaces to sets, is defined by

$$F(T) = SW(E^m,T) \qquad \text{for any linear space T}$$

$$\begin{cases} F(t) : F(T) \to F(H) \\ F(t){:}\,\Phi \mapsto t\circ\Phi \end{cases} \qquad \text{for any linear } \ t : \mathrm{T} \to \mathrm{H}$$

Theorem 2.2 *For any r-swlin mapping* $\Psi : E^m \to H$ *there exists a unique linear mapping* $f : E * \cdots * E \to H$ *such that*

$$f(x_1 * \cdots * x_m) = \Psi(x_1,\dots,x_m)$$

That is, the mapping $* : E^m \to T$ is an universal element for the functor F.

Proof. The proof generalizes to swlin maps the classical proof of universality of the tensor product, see [4], [6].

Uniqueness. Suppose that $* : E^m \to T$ and $\tilde{*} : E^m \to \tilde{T}$ are universal elements for the functor F, then, there exist linear maps

$$f:T\to\tilde{T} \qquad \text{and} \qquad g:\tilde{T}\to T$$

suchthat

$$f(x_1*\cdots*x_m)=x_1\,\tilde{*}\cdots\tilde{*}\,x_m \qquad \text{and} \qquad g(x_1\,\tilde{*}\cdots\tilde{*}\,x_m)=x_1*\cdots*x_m$$

thatis

$$gf(x_1*\cdots*x_m)=x_1*\cdots*x_m \qquad \text{and} \qquad fg(x_1\,\tilde{*}\cdots\tilde{*}\,x_m)=x_1\,\tilde{*}\cdots\tilde{*}\,x_m$$

by the universality of $*$ and $\tilde{*}$ it follows, respectively

$$1_T=g\circ f \qquad \text{and} \qquad 1_{\tilde{T}}=f\circ g$$

thus f and g are inverse linear isomorphisms.

Existence: Consider the free vector space $C(E^r)$ generated by the space E^r. Denote by $N(E^r)$ the subspace of $C(E^r)$ spanned by the vectors

$$\begin{aligned}&(x_\nu^{\mu_1}e_\nu,\ldots,\delta_1 y_1+\delta_2 y_2,\ldots,x_\nu^{\mu_r}e_\nu)-\delta_1(x_\nu^{\mu_1}e_\nu,\ldots,y_1,\ldots,x_\nu^{\mu_r}e_\nu)\\&-\delta_2(x_\nu^{\mu_1}e_\nu,\ldots y_2,\ldots,x_\nu^{\mu_r}e_\nu)\end{aligned}$$

for $\mu=\mu_1,\ldots,\mu_r\in I_r^m,\ \nu=\nu_1,\ldots,\nu_r\in I_r^n,\ \delta_i\in\Gamma$ and $x_\nu^{\mu_r}e_\nu, y_1, y_2\in E^r$.

Set $S=C(E^r)/N(E^r)$ and let $\pi:C(E^r)\to S$ be the canonical projection. Define the map

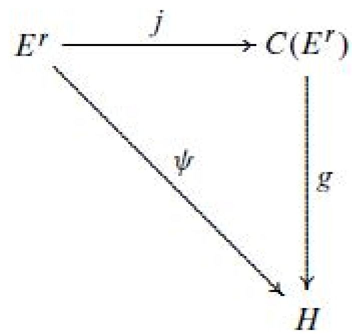
$$\begin{cases}*: & E^m\to S\\ *: & (x_1,\ldots,x_m)\mapsto\sum_{\mu,\nu}\lambda_\nu^\mu\pi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)\end{cases}$$

Since π is a homomorphism, it follows that $*$ is an r-swlin map.
If $z\in S$, then it is a finite sum

$$\begin{aligned}z&=\sum_\tau\delta^\tau\Big(\sum_{\mu,\nu}\lambda_\nu^\mu\pi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)\Big)_\tau\\&=\sum_\tau\delta^\tau(x_1*\cdots*x_m)_\tau\end{aligned}$$

so $\forall z\in S$, z is spanned by the products $x_1*\cdots*x_m$ and $I_m*=S$.

Moreover let $\psi:\mathrm{E}^r\to H$ be a r-linear map. Since $C(E^r)$ is a free vector space, there exists an unique linear map g such that the following diagram commutes



where j is the insertion of E^r in $C(E^r)$. So

$$g(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)=\psi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)$$

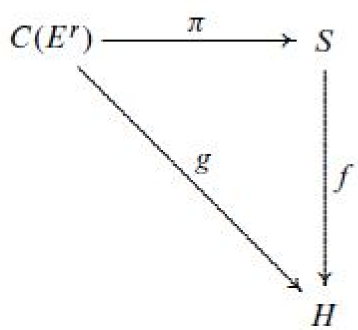
If

$$z=(x_\nu^{\mu_1}e_\nu,\ldots,\delta_1 y_1+\delta_2 y_2,\ldots,x_\nu^{\mu_r}e_\nu)-\delta_1(x_\nu^{\mu_1}e_\nu,\ldots,y_1,\ldots,x_\nu^{\mu_r}e_\nu)-\delta_2(x_\nu^{\mu_1}e_\nu,\ldots y_2,\ldots,x_\nu^{\mu_r}e_\nu)$$

Is a generator of $N(E^r)$, then

$$\begin{aligned}g(z)=\psi(z)&=\psi(x_\nu^{\mu_1}e_\nu,\ldots,\delta_1 y_1+\delta_2 y_2,\ldots,x_\nu^{\mu_r}e_\nu)-\delta_1\psi(x_\nu^{\mu_1}e_\nu,\ldots,y_1,\ldots,x_\nu^{\mu_r}e_\nu)-\delta_2\psi(x_\nu^{\mu_1}e_\nu,\ldots y_2,\ldots,x_\nu^{\mu_r}e_\nu)\\&=0\end{aligned}$$

then $N(E^r)\subseteq Ker\, g$. For the principal theorem on factor spaces, see [5], there exists an unique linear map f such that the following diagram commutes



that is, π is an universal element. So

$$\begin{aligned}(f\circ *)(x_1,\ldots,x_m)&=f(\sum_{\mu,\nu}\lambda_\nu^\mu\pi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu))\\&=\sum_{\mu,\nu}\lambda_\nu^\mu f\circ\pi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)\\&=\sum_{\mu,\nu}\lambda_\nu^\mu g(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)\end{aligned}$$

$$= \sum_{\mu,\nu} \lambda_\nu^\mu \psi(x_\nu^{\mu_1} e_\nu, \ldots, x_\nu^{\mu_r} e_\nu)$$
$$= \Psi(x_1, \ldots, x_m)$$

□

Example 2.4 *Consider the 2-swlin function* Φ *defined by*

$$\begin{cases} \Phi: & (\Re^2)^3 \to \Re \\ \Phi: & (x_1, x_2, x_3) \mapsto \lambda^{12}(x_1, x_2) + \lambda^{13}(x_1, x_3) + \lambda^{23}(x_2, x_3) \qquad \lambda^{12}, \lambda^{13}, \lambda^{23} \in \Re \end{cases}$$

where the bilinear function $(-,-)$, on the right side, is the inner product in $\Re^2$. By the theorem 2.2, the map $*:(\Re^2)^3 \to \Re^2 * \Re^2 * \Re^2$ is universal, so an unique linear function $f: \Re^2 * \Re^2 * \Re^2 \to \Re$ exists such that $f(x_1 * x_2 * x_3) = \Phi(x_1, x_2, x_3)$. Since $\Re^2 * \Re^2 * \Re^2$ is free, the function f is determined by its values $f(x_1 * x_2 * x_3)$ on the free generators $x_1 * x_2 * x_3$.

Corollary 2.1 *For any r-swlin map* $\Phi: E^m \to T$

$$*(x_1, \ldots, x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu (x_\nu^{\mu_1} e_\nu \otimes \cdots \otimes x_\nu^{\mu_r} e_\nu)$$

Proof. Since $\pi(x_\nu^{\mu_1} e_\nu, \cdots, x_\nu^{\mu_r} e_\nu) = x_\nu^{\mu_1} e_\nu \otimes \cdots \otimes x_\nu^{\mu_r} e_\nu$, by the theorem 2.2

$$\Phi(x_1, \ldots, x_m) = (f \circ *)((x_1, \ldots, x_m) = f(\sum_{\mu,\nu} \lambda_\nu^\mu (x_\nu^{\mu_1} e_\nu \otimes \cdots \otimes x_\nu^{\mu_r} e_\nu)$$

Example 2.5 *Let* $\Phi: (\Gamma^n)^n \to T$ *be a 2-swlin map. The tensor product* $\otimes: \Gamma^n \times \Gamma^n \to M^{n \times n}$ *is defined by* $x_{i_1} \otimes x_{i_2} = x_{i_1} x'_{i_2}$, $x_i \in \Gamma^n$, *see [4], then* $*: (\Gamma^n)^n \to \Gamma^n * \cdots * \Gamma^n$ *is given by*

$$x_1 * \cdots * x_n = \sum_{(i_1, i_2) \in I_2^n} \lambda^{(i_1, i_2)} x_{i_1} \otimes x_{i_2}$$
$$= \begin{pmatrix} \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{1i_1} x_{1i_2} & \cdots & \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{1i_1} x_{ni_2} \\ \cdots & \cdots & \cdots \\ \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{ni_1} x_{1i_2} & \cdots & \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{ni_1} x_{ni_2} \end{pmatrix}$$

3. $\{r, \lambda\}$ - determinant

If σ is a permutation, $\sigma \in S_r$, then the mapping $\sigma\phi: \Xi^r \to F$ is defined by $\sigma\phi(x_1, \ldots, x_r) = \phi(x_{\sigma_1}, \ldots, x_{\sigma_r})$. More generally

Definition 3.1 *Let* $\Phi(x_1,\ldots,x_m)$ *be an r-swlin map, for any permutation* $\sigma\in S_r$, *the mapping* $\sigma\Phi: E^m \to T$, *is defined by*

$$\sigma\Phi(x_1,\ldots,x_m)=\sum_{\mu,\nu}\lambda_\nu^\mu\sigma\phi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)=\sum_{\mu,\nu}\lambda_\nu^\mu\phi(x_\nu^{\sigma(\mu_1)}e_\nu,\ldots,x_\nu^{\sigma(\mu_r)}e_\nu)$$

Definition 3.2 *An r-swlin map* $\Phi(x_1,\ldots,x_m)$ *is said skewsymmetric if for any* $\sigma\in S_r$ *is* $\sigma\Phi=\varepsilon_\sigma\Phi$ *where* $\varepsilon_\sigma=1\,(\varepsilon_\sigma=-1)$ *for any even (odd) permutation* σ.

Theorem 3.1 *An r-swlin map* $\Phi=\sum\lambda_\nu^\mu\phi$ *is skewsymmetric if and only if* ϕ *is skewsymmetric. Proof.*Suppose ϕ skewsymmetric, then

$$\sigma\Phi=\sum_{\mu,\nu}\lambda_\nu^\mu\sigma\phi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)=\sum_{\mu,\nu}\lambda_\nu^\mu\varepsilon_\sigma\phi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)=\varepsilon_\sigma\Phi$$

Conversely, $\sigma\Phi=\varepsilon_\sigma\Phi$ implies

$$\sum_{\mu,\nu}\lambda_\nu^\mu\sigma\phi=\sum_{\mu,\nu}\lambda_\nu^\mu\varepsilon_\sigma\phi$$

so $\sum_{\mu,\nu}\lambda_\nu^\mu(\sigma\phi-\varepsilon_\sigma\phi)=0$ for all $x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu$, then $\sigma\phi=\varepsilon_\sigma\phi$. □

Theorem 3.2 *Every r-swlin map* $\Phi(x_1,\ldots,x_m)$ *determines an r-swlinskewsymmetric map* Ψ, *given by*

$$\Psi=\sum_\sigma\varepsilon_\sigma\sigma\Phi=\sum_{\mu,\nu}\sum_\sigma\lambda_\nu^\mu\varepsilon_\sigma\ \sigma\phi(x_\nu^{\mu_1}e_\nu,\ldots,x_\nu^{\mu_r}e_\nu)$$

where the second sum on right side is over all permutations $\sigma\in S_r$.

Proof. For any $\tau\in S_r$

$$\tau\Psi=\sum_{\mu,\nu}\tau(\sum_\sigma\lambda_\nu^\mu\varepsilon_\sigma\sigma\phi)=\sum_{\mu,\nu}\varepsilon_\tau(\sum_\sigma\lambda_\nu^\mu\varepsilon_\sigma\sigma\phi)=\varepsilon_\tau(\sum_{\mu,\nu}\sum_\sigma\lambda_\nu^\mu\varepsilon_\sigma\sigma\phi)=\varepsilon_\tau\Psi.$$

□

Theorem 3.3 *Let* $\Phi=\sum_{\mu,\nu}\lambda_\nu^\mu\phi: E^m\to F$ *be an r-swlinskewsymmetric map, then* Φ *is completely determined by its values on a basis of* E *and by the constants* λ_ν^μ.

Proof. Let $\{e_\nu\}$ be a basis of E. Let $x^i=\sum_{\xi=1}^n x_\xi^i e_\xi$, $i=1,\ldots,m$ be vectors in E and $X=(x_\xi^i)$, then

$$\Phi(x_1,\ldots,x_m)=\Phi(\sum_{\xi=1}^n x_\xi^1 e_\xi,\ldots,\sum_{\xi=1}^n x_\xi^m e_\xi)$$

$$= \sum_{\mu,\nu} \lambda_\nu^\mu \phi((\sum_{\xi=1}^n x_\xi^{\mu_1} e_\xi)_\nu, \ldots, (\sum_{\xi=1}^n x_\xi^{\mu_r} e_\xi)_\nu) \qquad \nu \in I_r^n,\, \mu \in I_r^m$$
$$= \sum_{\mu,\nu} \lambda_\nu^\mu (\sum_{\rho=\rho_1,\ldots,\rho_r} \varepsilon_\rho x_{\nu_{\rho_1}}^{\mu_1} \cdots x_{\nu_{\rho_r}}^{\mu_r} \phi(e_{\nu_{\rho_1}}, \ldots, e_{\nu_{\rho_1}})) \qquad \rho \in S_r$$
$$= \sum_{\mu,\nu} \lambda_\nu^\mu \mid X_\nu^\mu \mid \phi(e_{\nu_1}, \ldots, e_{\nu_r})$$

where X_ν^μ is the square submatrix of X determined by rows indexed by ν and columns indexed by μ.

Example 3.1 *Let* $\Phi : (\Re^3)^3 \to \Re^3$ *be a 2-swlin skewsymmetric map defined by*

$$\Phi(x_1, x_2, x_3) = \sum_{(i_1,i_2),(j_1,j_2) \in I_2^3} \lambda_{i_1,i_2}^{j_1,j_2} \phi \begin{pmatrix} x_{i_1,j_1} & x_{i_1,j_2} \\ x_{i_2,j_1} & x_{i_2,j_2} \end{pmatrix}$$

where $x_i = \sum_{k=1}^3 x_{k,i} e_k \in \Re^3$. Then

$$\Phi(x_1, x_2, x_3) = \sum_{(i_1,i_2),(j_1,j_2)\in I_2^3} \lambda_{i_1,i_2}^{j_1,j_2} \phi(x_{i_1 j_1} e_{i_1} + x_{i_2 j_1} e_{i_2}, x_{i_1 j_2} e_{i_1} + x_{i_2 j_2} e_{i_2})$$

$$= \sum_{(i_1,i_2),(j_1,j_2)\in I_2^3} \lambda_{i_1,i_2}^{j_1,j_2} \phi \begin{vmatrix} x_{i_1,j_1} & x_{i_1,j_2} \\ x_{i_2,j_1} & x_{i_2,j_2} \end{vmatrix} \phi(e_{i_1}, e_{i_2})$$

Definition 3.3 *Let* $\{e_\nu\}$ *be a basis of* E, *then an r-swlinskewsymmetric map* $\Delta_E(x_1, \ldots, x_m) : E^m \to \Gamma$ *such that* $\phi(e_{\nu_1}, \ldots, e_{\nu_r}) = 1,\ \nu \in I_r^n$, *is said an r-determinant function.*

The scalar $det_{r,\lambda} X = \sum_{\mu,\nu} \lambda_\nu^\mu \mid X_\nu^\mu \mid$ will be said the (r, λ)-determinant of $X = (x_\xi^i)$, relative to the basis $\{e_\nu\}$. If $\lambda_\nu^\mu = \mid X_\nu^\mu \mid$ we denote $det_r X = \mid X \mid_r = \sum_{\mu,\nu} \mid X_\nu^\mu \mid^2$, see [2].

Example 3.2 *In order to obtain a non-trivial example of r-determinant function, consider a 2-swlin function* $\Phi = \sum_{\mu,\nu} \lambda_\nu^\mu \phi$ *defined by*

$$\Phi(x_1, \ldots, x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu \langle e^{*\mu_1}, x_\nu^{\mu_1} e_\nu \rangle \cdots \langle e^{*\mu_r}, x_\nu^{\mu_r} e_\nu \rangle$$

thatis

$$\phi(x_\nu^{\mu_1} e_\nu, \ldots, x_\nu^{\mu_r} e_\nu) = \langle e^{*\mu_1}, x_\nu^{\mu_1} e_\nu \rangle \cdots \langle e^{*\mu_r}, x_\nu^{\mu_r} e_\nu \rangle$$

where $\{e_\nu\}, \{e^{*\nu}\}$ are a pair of dual bases in E and $E^* = L(E) = \{f : f : E \to \Gamma, f \quad \text{linear}\}$ respectively, with $dimE = dimE^* \geq r$. The bilinear function $\langle , \rangle$ is non-degenerate and it is defined by

$$\langle e^{*\mu_i}, x_\nu^{\mu_i} e_\nu \rangle = e^{*\mu_i}(x_\nu^{\mu_i} e_\nu)$$

then

$$\Phi(x_1, \ldots, x_m) = \sum_\mu \lambda_\mu^\mu \langle e^{*\mu_1}, x_{\mu_1}^{\mu_1} e_{\mu_1} \rangle \cdots \langle e^{*\mu_r}, x_{\mu_r}^{\mu_r} e_{\mu_r} \rangle$$
$$= \sum_\mu^\mu \lambda_\mu^\mu x_{\mu_1}^{\mu_1} \cdots x_{\mu_r}^{\mu_r}$$

The set of the r-swlin maps is denoted by $SW(E^m, T)$. The exponential functor F, from linear spaces to sets, is defined by

$$F(T) = SW(E^m, T) \qquad \text{for any linear space T}$$
$$\begin{cases} F(t) : F(T) \to F(H) \\ F(t) : \Phi \mapsto t\Phi \end{cases} \qquad \text{for any linear t : T} \to \text{H}$$

The following proposition states the universality of the r-determinant function.

Theorem 3.4 *Let $\Delta_E = \sum_{\mu,\nu} \lambda_\nu^\mu \phi : E^m \to \Gamma$ be an r-determinant function in E, then for any r-swlinskewsymmetric mapping $\Theta = \sum_{\mu,\nu} \lambda_\nu^\mu \theta : E^m \to F$, there is an unique vector $f \in F$ such that*

$$\Theta(x_1, \ldots, x_m) = (\Delta_E(x_1, \ldots, x_m)(f) = \sum_{\mu,\nu} \lambda_\nu^\mu \mid X_\nu^\mu \mid f_\nu \qquad \mu \in I_r^m, \ \nu \in I_r^n, \ x_i \in E$$

where f_ν are the components of the vector

$$f = (\theta(e_{\nu_1^1}, \ldots, e_{\nu_r^1}), \ldots, \theta(e_{\nu_1^{\binom{n}{r}}}, \ldots, e_{\nu_r^{\binom{n}{r}}}))$$

and ν^i are the $\binom{n}{r}$ elements of I_r^n.

Proof. Let $\{e_i\}, \ i = 1, \ldots, n$ be a basis of E such that

$$\Delta_E(x_1, \ldots, x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu \mid X_\nu^\mu \mid \phi(e_{\nu_1}, \ldots, e_{\nu_r}) = \sum_{\mu,\nu} \lambda_\nu^\mu \mid X_\nu^\mu \mid$$

that is , $\phi(e_{\nu_1}, \ldots, e_{\nu_r}) = 1$.

Then, for any r-swlin skew symmetric map

$$\Psi(x_1,\ldots,x_m)=\sum_{\mu,\nu}\lambda_\nu^\mu \mid X_\nu^\mu \mid \psi=(\Delta_E(x_1,\ldots,x_m))(f)$$

itfollows

$$\psi(e_{\nu_1},\ldots,e_{\nu_r})=\phi(e_{\nu_1},\ldots,e_{\nu_r})\theta(e_{\nu_1},\ldots,e_{\nu_r})=1\cdot\theta(e_{\nu_1},\ldots,e_{\nu_r})$$

so Θ and Ψ have the same values on the basis $\{e_\nu\}$ and by theorem 3.3 it follows $\Theta=\Psi$. □

If Δ_E and $\Delta_E^{'}$ are two r-determinant functions in E, then $\eta\Delta_E+\theta\Delta_E^{'},\ \eta,\theta\in\Gamma$, is a r-determinant function too.

Let Δ_F be an r-determinant function in F and let $\psi:E\to F$ be a linear mapping of vector spaces, where $dimE=n, dimF=t$, then $\Delta_\psi:E^m\to\Gamma$, defined by

$$\Delta_\psi(x_1,\ldots,x_m)=\Delta_F(\psi x_1,\ldots,\psi x_m)=\sum_{\mu,\tau}\lambda_\tau^\mu\phi_F((\psi x^{\mu_1})_\tau,\ldots,(\psi x^{\mu_r})_\tau)$$

is an r-determinant function in E, where $\phi_F:F^r\to\Gamma$ is an r-linear mapping on F, $\mu\in I_r^m,\ \tau\in I_r^t$.

By theorem 3.4, $\Delta_\psi=\Delta_F(f)=\sum_{\mu,\nu,\tau}\lambda_\tau^\mu \mid X_\nu^\tau \mid f_\nu$ for an unique vector $f=(f_\nu)$.

Let $\Delta_F^{'}$ be another nonnullswilin skew symmetric map, then

$$\Delta_F^{'}=\Delta_F(g)=\sum_{\mu,\nu,\tau}\lambda_\tau^\mu \mid X_\nu^\tau \mid g_\nu$$

and

$$\Delta_\psi^{'}=\Delta_\psi(g)=(\Delta_F(f))(g)=\sum_{\mu,\nu,\tau}\lambda_\tau^\mu \mid X_\nu^\tau \mid f_\nu g_\nu=\Delta_F^{'}(f_\nu)$$

so the vector f does not depend on the choise of Δ_F and it is determined by the map ψ, then the notation $f=det\psi$.

Example 3.3 *Let ψ and A_ψ be a linear map and its matrix respectively, defined by*

$$\begin{cases}\psi:\Re^2\to\Re^3\\ \psi:(x,y)\mapsto(x,y,x+y)\end{cases}\qquad A_\psi=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}$$

besides let $\Delta_{\Re^3}:(\Re^3)^3\to\Re$ be a 2-determinant function and $x_i\in\Re^2$, then

$$\Delta_{\psi} = \Delta_{\Re^3}(\psi x_1, \psi x_2, \psi x_3) = \lambda^{12}\phi(\psi x_1, \psi x_2) + \lambda^{13}\phi(\psi x_1, \psi x_3) + \lambda^{23}\phi(\psi x_2, \psi x_3)$$

$$= \lambda^{12}\phi(\sum_{i=1}^{2} x_{i1}\psi e_i, \sum_{i=1}^{2} x_{i2}\psi e_i) + \lambda^{13}\phi(\sum_{i=1}^{2} x_{i1}\psi e_i, \sum_{i=1}^{2} x_{i3}\psi e_i) \quad + \lambda^{23}\phi(\sum_{i=1}^{2} x_{i2}\psi e_i, \sum_{i=1}^{2} x_{i3}\psi e_i)$$

$$= \lambda^{12} \,|\, X^{12} \,|\, \phi(\psi e_1, \psi e_2) + \lambda^{13} \,|\, X^{13} \,|\, \phi(\psi e_1, \psi e_2) + \lambda^{23} \,|\, X^{23} \,|\, \phi(\psi e_1, \psi e_2)$$

where $| X^{ij} | = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix}$. Since

$$\phi(\psi e_1, \psi e_2) = \phi((1,0,1),(0,1,1)) = \lambda_{12}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \lambda_{13}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \lambda_{23}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = \lambda_{12} + \lambda_{13} - \lambda_{23}$$

then

$$\Delta_{\psi} = \lambda^{12} \,|\, X^{12} \,|\, det_{2,\lambda}\psi + \lambda^{13} \,|\, X^{13} \,|\, det_{2,\lambda}\psi + \lambda^{23} \,|\, X^{23} \,|\, det_{2,\lambda}\psi = \Delta_{\Re^3}(det_{2,\lambda}\psi)$$

The expression for $det\psi$ may be obtained immediately by the matrix A_{ψ}, see [2]

$$det_{2,\lambda} A_{\psi} = det_{2,\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \lambda_{12}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \lambda_{13}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \lambda_{23}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = \lambda_{12} + \lambda_{13} - \lambda_{23}$$

Theorem 3.5 *Let $\psi : E \to F$ be a linear mapping and $A_{\psi} = (\alpha_{\nu}^{\tau})$ its matrix relative to the bases $\{e_{\nu}\}, \{f_{\tau}\}$, $\nu = 1,\ldots,n$, $\tau = 1,\ldots,t$. Let $\Delta_F = \sum_{\mu,\tau} \lambda_{\tau}^{\mu}\phi_F : F^m \to \Gamma$ be an r-determinant function. If $\phi_F(f_{\tau}^{\mu_1},\ldots, f_{\tau}^{\mu_r}) = 1$, then*

i)
$$\Delta_{\psi}(x_1,\ldots,x_m) = \sum_{\mu,\tau} \lambda_{\tau}^{\mu}(\sum_{\nu} | X_{\nu}^{\mu} \,||\, A_{\nu}^{\tau} |) \quad \mu \in I_r^m, \nu \in I_r^n, \tau \in I_r^t$$

ii)
$$\Delta_{\psi}(e_1,\ldots,e_n) = \sum_{\nu,\tau} \lambda_{\tau}^{\nu} \,|\, A_{\nu}^{\tau} \,|$$

where A_{ν}^{τ} is the submatrix of A determined by rows indexed by ν and columns indexed by τ, for $\nu = \nu_1,\ldots,\nu_r \in I_r^n, \tau = \tau_1,\ldots,\tau_r \in I_r^t$. The vectors $x_1,\ldots,x_m$, relative to the basis $\{e_{\nu}\}$, are expressed by $x^{\mu} = \sum_{\nu=1}^{n} x_{\nu}^{\mu} e_{\nu}$, $\mu = 1,\ldots,m$ and $X = (x_{\nu}^{\mu})$.

Proof. i)

$$\Delta_\psi(x_1,\dots,x_m)=\Delta_F(\psi x_1,\dots,\psi x_m)=\Delta_F(\sum_{\nu=1}^{n}x_\nu^1\psi e_\nu,\dots,\sum_{\nu=1}^{n}x_\nu^m\psi e_\nu)$$
$$=\Delta_F(\sum_{\nu=1}^{n}x_\nu^1\sum_{\tau=1}^{t}\alpha_1^\tau f_\tau,\dots,\sum_{\nu=1}^{n}x_\nu^m\sum_{\tau=1}^{t}\alpha_m^\tau f_\tau)$$
$$=\Delta_F(\sum_{\tau=1}^{t}(\sum_{\nu=1}^{n}x_\nu^1\alpha_\nu^\tau)f_\tau,\dots,\sum_{\tau=1}^{t}(\sum_{\nu=1}^{n}x_\nu^m\alpha_\nu^\tau)f_\tau)$$
$$=\sum_{\mu,\tau}\lambda_\tau^\mu\phi_F(((\sum_{\nu=1}^{n}x_\nu^{\mu_1}\alpha_\nu^\tau)f_\tau),\dots,((\sum_{\nu=1}^{n}x_\nu^{\mu_r}\alpha_\nu^\tau)f_\tau)\qquad \tau\in I_r^t,\ \mu\in I_r^m$$
$$=\sum_{\mu,\tau}\lambda_\tau^\mu(\sum_{\rho=\rho_1,\dots,\rho_r}\varepsilon_\rho(\sum_{\nu=1}^{n}x_\nu^{\mu_1}\alpha_\nu^{\tau_{\rho_1}})\cdots(\sum_{\nu=1}^{n}x_\nu^{\mu_r}\alpha_\nu^{\tau_{\rho_r}}))\phi_F(f_\tau^{\rho_1},\dots,f_\tau^{\rho_r})$$

$\rho\in S_r$, by

$$\sum_{\rho=\rho_1,\dots,\rho_r}\varepsilon_\rho(\sum_{\nu=1}^{n}x_\nu^{\mu_1}\alpha_\nu^{\tau_{\rho_1}})\cdots(\sum_{\nu=1}^{n}x_\nu^{\mu_r}\alpha_\nu^{\tau_{\rho_r}})=\sum_\nu|X_\nu^\mu||A_\nu^\tau|$$ it follows i).

ii) It is a special case of i) for $X=I_n$.

The scalar $det_{r,\lambda}\psi=\sum_{\mu,\nu}\lambda_\nu^\mu|A_\nu^\mu|$ will be called the (r,λ)-determinant of ψ, relative to the bases $\{e_\nu\},\{f_\mu\}$. If $\lambda_\nu^\mu=|A_\nu^\mu|$, then $\sum_{\mu,\nu}|A_\nu^\mu|^2$ will be denoted by $det_r\psi$ or $|\psi|_r$ □

Theorem 3.6 *Let $\psi:E\to F$ and $\theta:F\to G$ be linear mappings of vector spaces. Let Δ_F be a determinant function in F. If $x_1,\dots,x_m$ are vectors in E, then*

$$\Delta_{\theta\circ\psi}(x_1,\dots,x_m)=\Delta_\theta\circ\Delta_\psi(x_1,\dots,x_m)$$

Proof.
$$\Delta_{\theta\circ\psi}(x_1,\dots,x_m)=\Delta_G(\theta\circ\psi(x_1,\dots,x_m))=\Delta_\theta(\psi(x_1,\dots,\psi x_m))=\Delta_\theta\circ\Delta_\psi(x_1,\dots,x_m)$$
□

4. The (t,k)-forms

Let $\Re_p^n$ be the tangent space of $\Re^n$ at the point p and let $(\Re_p^n)^*$ be the dual space. Let $\Lambda^k(\Re_p^n)^*$ be the linear space of the k-linear alternating maps $\phi:(\Re_p^n)^k\to\Re$, then denote by $\Lambda_t^k(\Re_p^n)^*$, with $k\le t\le n$, the set of all k-linear alternating maps $\phi:(\Re_p^n)^t\to\Re$. The set $\Lambda_t^k(\Re_p^n)^*$, by the usual operations of functions, is a linear space. If $\phi_1,\dots,\phi_t$ belong to $(\Re_p^n)^*$, then an element $\phi_1\wedge\dots\wedge\phi_t\in\Lambda_t^k(\Re_p^n)^*$ is obtained by setting

$$(\phi_1 \wedge \ldots \wedge \phi_t)(v_1, \ldots, v_k) = det_{k,\lambda}\phi_i(v_j) = \begin{vmatrix} \phi_1(v_1) & \cdots & \phi_1(v_k) \\ \cdots & \cdots & \cdots \\ \phi_t(v_1) & \cdots & \phi_t(v_k) \end{vmatrix}$$

where $i = 1, \ldots, t$, $j = 1, \ldots, k$ and $v_j \in \Re^n$.

Observe that $\phi_1 \wedge \ldots \wedge \phi_t$ is k-linear and alternate.

Example 4.1 *When* ϕ_1, ϕ_2, ϕ_3 *belong to* $(\Re^3_p)^*$, *an element* $\phi_1 \wedge \phi_2 \wedge \phi_3 \in \Lambda^2_3(\Re^3_p)^*$ *is obtained by the 2-swlin skewsymmetric map*

$$(\phi_1 \wedge \phi_2 \wedge \phi_3)(v_1, v_2) = det_{2,\lambda}\phi_i(v_j) = \begin{vmatrix} \phi_1(v_1) & \phi_1(v_2) \\ \phi_2(v_1) & \phi_2(v_2) \\ \phi_3(v_1) & \phi_3(v_2) \end{vmatrix} = \sum_{i_1 < i_2} \lambda_{i_1 i_2} \begin{vmatrix} \phi_{i_1}(v_1) & \phi_{i_1}(v_2) \\ \phi_{i_2}(v_1) & \phi_{i_2}(v_2) \end{vmatrix}$$

$(i_1, i_2) \in I^3_2$, $\lambda_{i_1 i_2} \in \Re$

and $\phi_1 \wedge \phi_2 \wedge \phi_3$ is a bilinear alternating map on the vectors v_1, v_2.

Let $x^i : \Re^n \to \Re$ be the function which assigns to each point of $\Re^n$ its i^{th}-coordinate. Then $(dx^i)_p$ is a linear map in $(\Re^n)^*$ and the set $\{(dx^i)_p ; i = 1, \ldots, n\}$ is the dual basis of the standard $\{(e_i)_p\}$. The element $(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_t})_p$ is denoted by $(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p$ and belongs to $\Lambda^k_t(\Re^n_p)^*$.

Theorem 4.1 *The set* $\{(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p\}, i_1, \ldots, i_t \in I^n_t$ *is a basis for* $\Lambda^k_t(\Re^n_p)^*$. *Proof.* the elements of $\{(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p\}$ are linearly independent. In fact, suppose

$$\sum_{i_1, \ldots, i_t \in I^n_t} a_{i_1, \ldots, i_t} dx^{i_1} \wedge \cdots \wedge dx^{i_t} = 0$$

then, for any $(e_{j_1}, \ldots, e_{j_k})$, with $j_1, \ldots, j_k \in I^n_k$, it follows

$$\sum_{i_1, \ldots, i_t \in I^n_t} a_{i_1, \ldots, i_t} dx^{i_1} \wedge \cdots \wedge dx^{i_t}(e_{j_1}, \ldots, e_{j_k})$$

$$= \sum_{i_1, \ldots, i_t \in I^n_t} a_{i_1, \ldots, i_t} \begin{vmatrix} dx^{i_1} e_{j_1} & \cdots & dx^{i_1} e_{j_k} \\ \cdots & \cdots & \cdots \\ dx^{i_t} e_{j_1} & \cdots & dx^{i_t} e_{j_k} \end{vmatrix}$$

$$= \sum_{i_1,\dots,i_t \in I_t^n} a_{i_1,\dots,i_t} \begin{vmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_k}^{i_1} \\ \cdots & \cdots & \cdots \\ \delta_{j_1}^{i_t} & \cdots & \delta_{j_k}^{i_t} \end{vmatrix}$$

$$= \sum_{r_1,\dots,r_t} \lambda_{r_1,\dots,r_t} a_{r_1,\dots,r_t} \qquad r_1,\dots,r_t \in (I_t^n)_{j_1,\dots,j_k}$$

$$= 0$$

Without loss of generality, suppose $\lambda_{r_1,\dots,r_t}$ all equal, then the $\binom{n}{k}$ equations $\sum_{r_1,\dots,r_t} a_{r_1,\dots,r_t} = 0,\ r_1,\dots,r_t \in (I_t^n)_{j_1,\dots,j_k},\ j_1,\dots,j_k \in I_k^n$, are a linear omogeneous full rank system, so it has only the trivial solution. That is $a_{i_1,\dots,i_t} = 0$.

The set $\{(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p\}$ spans $\Lambda_t^k(\Re_p^n)^*$, in other words any $\phi \in \Lambda_t^k(\Re_p^n)^*$ may be written

$$\phi = \sum_{i_1,\dots,i_t \in I_t^n} a_{i_1,\dots,i_t} dx^{i_1} \wedge \cdots \wedge dx^{i_t} \qquad i_1,\dots,i_t \in I_t^n$$

in fact, if

$$\psi = \sum_{i_1,\dots,i_t \in I_t^n} \phi(e_{i_1},\dots,e_{i_t}) dx^{i_1} \wedge \cdots \wedge dx^{i_t}$$

then $\psi(e_{i_1},\dots,e_{i_t}) = \phi(e_{i_1},\dots,e_{i_t})$ for all $i_1,\dots,i_t \in I_t^n$, so $\psi = \phi$. Setting $\psi(e_{i_1},\dots,e_{i_t}) = a_{i_1,\dots,i_t}$, it follows the expression of ϕ. □

The above proposition generalizes the known theorem about the basis $\{dx^{i_1} \wedge \cdots \wedge dx^{i_k}\}$ of the space $\Lambda^k(\Re_p^n)^*$, see [1].

Theorem 4.2 *The linear spaces* $\Lambda_t^k(\Re_p^n)^*$ *and* $\Lambda^k(\Re_p^n)^*$ *coincide.*

Proof. Let $\omega = (\phi_1 \wedge \cdots \wedge \phi_t)(v_1,\dots,v_k) \in \Lambda_t^k(\Re_p^n)^*$, then

$$\omega = \sum_{i_1,\dots,i_k \in I_k^n} \lambda_{i_1,\dots,i_k} \begin{vmatrix} \phi_{i_1}(v_1) & \cdots & \phi_{i_1}(v_k) \\ \cdots & \cdots & \cdots \\ \phi_{i_k}(v_1) & \cdots & \phi_{i_k}(v_k) \end{vmatrix} = \sum_{i_1,\dots,i_k \in I_k^n} \lambda_{i_1,\dots,i_k} (\phi_1 \wedge \cdots \wedge \phi_k)(v_1,\dots,v_k)$$

so $\omega \in \Lambda^k(\Re_p^n)^*$. Conversely, let 0 be the null function in $(\Re_p^n)^*$, then any $\psi \in \Lambda^k(\Re_p^n)^*$ may be written as

$$\psi = (\psi_1 \wedge \cdots \wedge \psi_k)(v_1,\ldots,v_k) = (\psi_1 \wedge \cdots \wedge \psi_k \wedge 0 \wedge \ldots \wedge 0)(v_1,\ldots,v_k) \text{ so } \psi \in \Lambda_t^k(\Re_p^n)^*.$$

If $\omega \in \Lambda_t^k(\Re_p^n)^*$, then ω may be decomposed by elements of $\Lambda_{t-j}^k(\Re_p^n)^*$, where $k \le t-j \le t$, in fact

Theorem 4.3 *Let* $\omega = (\phi_1 \wedge \ldots \wedge \phi_t)(v_1,\ldots,v_k) \in \Lambda_t^k(\Re_p^n)^*$, *then*

$$\omega = \frac{\lambda_{i_1,\ldots,i_{t-j}}}{(t-k)\cdots(t-k-j+1)} \sum_{I_{t-j}^t} (\phi_{i_1} \wedge \ldots \wedge \phi_{i_{t-j}})(v_1,\ldots,v_k)$$

Proof.

$$\begin{aligned}\omega &= \frac{\lambda_{i_1,\ldots,i_{t-1}}}{(t-k)} \sum_{I_{t-1}^t} (\phi_{i_1} \wedge \ldots \wedge \phi_{i_{t-1}})(v_1,\ldots,v_k) \\ &= \cdots \qquad \cdots \qquad \cdots \\ &= \frac{\lambda_{i_1,\ldots,i_{t-j}}}{(t-k)\cdots(t-k-j+1)} \sum_{I_{t-j}^t} (\phi_{i_1} \wedge \ldots \wedge \phi_{i_{t-j}})(v_1,\ldots,v_k)\end{aligned}$$

indeed ω is the sum of $\binom{t}{k}$ determinants, the last right side has the same number

$$\frac{t\cdots(t-j+2)}{(t-k)\cdots(t-k-j+1)}\binom{t-j}{k}\binom{t-j+1}{t-j}$$

□

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