American Review of Mathematics and Statistics
June 2016, Vol. 4, No. 1, pp. 31-44
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v4n1a4
URL: https://doi.org/10.15640/arms.v4n1a4

Iterated Defect Correction with B-Splines for a Class of Strongly Nonlinear Two-Point Boundary Value Problems

Sennur SOMALI[1] & Ayse BELER[2]

Abstract

We consider numerical solution for a class of strongly nonlinear two-point boundary value problem. By giving the theoretical and numerical results, we discuss the high order convergence behavior of the iterated defect correction technique based on implicit trapezoid method with B-splines for the problems. This method provides high accuracy results for the solution of Troesch's problem for large parameter λ

Keywords: Strongly nonlinear two-point boundary value problems; Iterated defect correction; B-spline; Polynomial approximation; Troesch's problem

1. Introduction

One important class of second order nonlinear differential equations is related to some heat-conduction problems and diffusion problems. In this paper, we study the following strongly nonlinear two-point boundary value problem

$$\begin{gathered} -\frac{d}{dx}\left(k(y)\frac{dy}{dx}\right) = f(x,y), \qquad 0 \le x \le 1 \\ y(0) = \alpha, \quad y(1) = \beta, \end{gathered} \tag{1}$$

where α and β are given constants in $I \subset \mathbb{R}, D = ([0,1] \times \mathbb{R} \subset \mathbb{R}^2), f \in C^{2m+4}(D), k(y) \in C^{2m+4}(I), m \in \mathbb{Z}^+$ and $k(y) > 0$ for all $y \in I$.

Since it is difficult to give the analytic solution of the problem (1), even if the function $f(x,y)$ is linear in y, various numerical methods have been developed to solve this problem. For example, we quote finite difference methods in [2],[5], Petrov-Galerkin method in [16], shooting methods in [1], [7], [21], [22], spline methods in [4], [18], [15], variation iteration methods [20], [6], collocation methods [9], asymptotic approximation [23] and Numeral's method in [24].

In [24], the problem (1) is discretized by fourth order Numerov's method and nonlinear monotone iterative algorithm is presented to compute the solutions of the resulting discrete problems. Some applications and numerical results are given to demonstrate high efficiency of the approach.

[1] Department of Mathematics, Faculty of Sciences, Dokuz Eylül University, Tınaztepe, Buca, 35160 Izmir, Turkey.
E-mail: sennur.somali@deu.edu.tr;
[2] Department of Mathematics, Faculty of Sciences, Dokuz Eylül University, Tınaztepe, Buca, 35160 Izmir, Turkey.
E-mail: ayse.beler@ogr.deu.edu.tr

In our study, we develop the Iterated Defect Correction (IDeC) technique by using B-spline interpolation of odd degree. The method of IDeC is one of the most powerful technique for the improvement of numerical solutions of initial and boundary value problems for ordinary differential equations. The idea behind the IDeC is carried out in the following way:

Compute a simple, basic approximation and form its defect with respect to the given differential equation by a piecewise interpolant. This defect is used to define a neighboring problem whose exact solution is known. Solving the neighboring problem with the basic discretization scheme yields a global error estimate. This can be used to construct an improved approximation, and the procedure can be iterated. IDeC methods originated from an idea of Zadunisky [25]. An asymptotic analysis $(h \to 0)$ of such an iterative procedure based on global error estimates is given by Frank [11, 12, 13]. In [14] Defect Correction for stiff differential equations, in [17] Mixed Defect Correction Iteration for the solution of a singular perturbation problem, in [19] an error analysis of Iterated Defect Correction methods for linear differential-algebraic equations, and in [10] the Iterated Defect Correction Methods for Singular two point boundary value problems are studied.

The outline of the paper is as follows: In Section2, the formulation of IDeC technique to the system of nonlinear two-point boundary value problem corresponding to (1) is given to use for improving the approximate solutions. We establish the asymptotic expansion of the global error for the implicit trapezoidal method in Section3. In Section4, we show that for an interpolating B-spline polynomial of degree $2m+1$ ($m \in \mathbb{Z}^+$) with the maximum IDeC step is m and the convergence is $O(h^{2m+2})$. In Section5, two test problems are presented by numerical results to verify the theoretical results in the previous chapter. Moreover, we present the efficiency of the IDeC method to apply the transformed form of Troesch's problem in [6]. The IDeC method provides the high accuracy result for large parameter $\lambda \geq 10$.

2. Application Of Iterated Defect Correction Techniques

Applying the transformation $z = k(y)\frac{dy}{dx}$ to (1), we obtain the following associated system of the first order

$$\begin{aligned} y' &= \quad g(y,z) \\ z' &= \quad -f(x,y), \end{aligned} \tag{2}$$

where $g(y,z) = \frac{z}{k(y)}$ with the boundary conditions$y(0) = \alpha, y(1) = \beta$ and $(y(x), z(x))^T$ denotes the exact solution for (2).

The problem (2) will be called as original boundary value problem (OP). The approximate solutions, $u_i^{[0]} \approx y(x_i)$ and $v_i^{[0]} \approx z(x_i)$ are obtained by implicit trapezoidal method which are based on the following difference schemes

$$\begin{aligned} u_i^{[0]} &= \quad u_{i-1}^{[0]} + \frac{h}{2}[g(u_{i-1}^{[0]}, v_{i-1}^{[0]}) + g(u_i^{[0]}, v_i^{[0]})], \\ v_i^{[0]} &= \quad v_{i-1}^{[0]} - \frac{h}{2}[f(x_{i-1}, u_{i-1}^{[0]}) + f(x_i, u_i^{[0]})], \end{aligned} \tag{3}$$

on the uniform grid $x_i = ih$ for $i = 0,1,\dots,N,$ with stepsize $h = 1/N$ and boundary conditions $u_0^{[0]} = \alpha, u_N^{[0]} = \beta$. The nonlinear system (3) is solved by Newton's method for $u_i^{[0]}$ and $v_i^{[0]}$. Mathematica has a built-in command to solve this nonlinear equation. We interpolate $u_i^{[0]}$ and $v_i^{[0]}$ by B-Spline piecewise polynomial functions $p^{[0]}(x)$ and $q^{[0]}(x)$ of fixed degree, $2m+1$ that satisfies the following conditions:

1. *Interpolation:*

$$p^{[0]}(x_i) = u_i^{[0]},$$
$$q^{[0]}(x_i) = v_i^{[0]}, \quad i = 0,1,\dots,N.$$

2. *Smoothness:*

$$\lim_{x\to x_i^-} p^{[0](k)}(x) = \lim_{x\to x_i^+} p^{[0](k)}(x), \quad \lim_{x\to x_i^-} q^{[0](k)}(x) = \lim_{x\to x_i^+} q^{[0](k)}(x)$$
$$\lim_{x\to x_i^-} q^{[0](k)}(x) = \lim_{x\to x_i^+} q^{[0](k)}(x), \quad k = 0,1,\dots,2m$$

3. *Interval of definition:* $p^{[0]}(x)$ and $q^{[0]}(x)$ is polynomial of degree at most $2m+1$ on each subinterval$[x_{i-1}, x_i]$.

The interpolations yield the defects when they substitute into (OP) (2)

$$d_y^{[0]}(x) = p^{[0]'}(x) - g(p^{[0]}(x), q^{[0]}(x))$$
$$d_z^{[0]}(x) = q^{[0]'}(x) + f(p^{[0]}(x)), q^{[0]}(x)).$$

By adding these defect terms to the right hand side of (OP) (2), we get a new BVP which is called by neighboring boundary value problem (NP) as

$$y' = g(y,z) + d_y^{[0]}(x),$$
$$z' = -f(x,y) + d_z^{[0]}(x). \tag{4}$$

Notice that; the exact solutions of (4), $p^{[0]}(x)$ and $q^{[0]}(x)$ are known. Then we solve (4) by the implicit trapezoidal method to obtain the numerical approximate solutions $\eta_i^{[0]} \approx p^{[0]}(x_i)$ and $\xi_i^{[0]} \approx q^{[0]}(x)$, $i = 0,1,\dots,N$ with

$$\eta_0^{[0]} = p^{[0]}(0) \quad \text{and} \quad \eta_N^{[0]} = p^{[0]}(1).$$

We can use the known global discretization errors $\eta_i^{[0]} - p^{[0]}(x_i)$ and $\xi_i^{[0]} - q^{[0]}(x_i)$ of (NP) (4) as an estimate for the unknown global discretization errors $u_i^{[0]} - y(x_i)$ and $v_i^{[0]} - z(x_i)$. The original idea of estimating global discretization error in this way is due to [25]. The improvement of our first solutions $u_i^{[0]}$ and $v_i^{[0]}$ is given by

$$u_i^{[1]} = u_i^{[0]} - (\eta_i^{[0]} - p^{[0]}(x_i)), \quad i = 1,2,\dots,N-1$$
$$v_i^{[1]} = v_i^{[0]} - (\xi_i^{[0]} - q^{[0]}(x_i)), \quad i = 0,1,\dots,N.$$

This procedure can be used iteratively as

$$u_i^{[j+1]} = u_i^{[0]} - (\eta_i^{[j]} - p^{[j]}(x_i)), \quad i = 1,2,\dots,N-1$$
$$v_i^{[j+1]} = v_i^{[0]} - (\xi_i^{[j]} - q^{[j]}(x_i)), \quad i = 0,1,\dots,N, \tag{5}$$

$j = 0,1,\dots,j_{\max}$, where j denotes the defect number.

3. Asymptotic Expansion of the Global Errors

The truncation error for the implicit trapezoid method $T_{h,i}$ is obtained as

$$T_{h,i} = \begin{bmatrix} \tau_{y,i} \\ \tau_{z,i} \end{bmatrix} = \sum_{k=2}^{K} \ell_{k+1}\, h^{k+1} \begin{bmatrix} y^{k+1}(x_i) \\ z^{k+1}(x_i) \end{bmatrix} + O(h^{K+1}), \qquad i = 1,2,\dots,N,$$

where $\ell_{k+1} = -\frac{(k-1)}{2(k+1)!}$.

The asymptotic expansions of the global errors of the implicit trapezoidal method applied to (OP)(2) and (NP) (4) are derived by using the same technique in Frank [13]. The asymptotic expansion of the global error for (OP) is as follows

$$U^{[0]}(x_i) - Y(x_i) = \sum_{n=1}^{m+1} h^{2n}\, E_{2n}(x_i) + \Delta_i, \tag{6}$$

where $\| \Delta_i \| = O(h^{2m+4})$,

$$U^{[0]}(x_i) = \begin{bmatrix} u^{[0]}(x_i) \\ v^{[0]}(x_i) \end{bmatrix}, \quad E_{2n}(x_i) = \begin{bmatrix} e_{y,2n}(x_i) \\ e_{z,2n}(x_i) \end{bmatrix}, \quad Y(x_i) = \begin{bmatrix} y(x_i) \\ z(x_i) \end{bmatrix},$$

and $E_{2n}(x)$ is the smooth solution to the following system of linear boundary value problem

$$\begin{aligned} E'_{2n}(x) &= J(Y)E_{2n}(x) + \ell_{2n+1}Y^{(2n+1)}(x) \\ &\quad + \text{const.}\, H_{2n}\big(F_2, F_4, \dots, F_{2n-2}, Y, Y', \dots, Y^{(2n-2)}, K_1, K_2\big) \end{aligned} \tag{7}$$

with the boundary conditions

$$M_0 E_{2n}(0) + M_1 E_{2n}(1) = 0,$$

$$\text{where } M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad J(Y) = \begin{bmatrix} g_y(y,z) & g_z(y,z) \\ -f_y(x,y) & 0 \end{bmatrix} \tag{8}$$

and F_{2L} is the function of $(E_{2L}, E'_{2L}, \dots, E_{2L}^{(2n-2L)})$ for $L = 1,2,\dots,2n-2$, K_1, K_2 are functions of partial derivatives of f and g of order $\leq 2n-1$, H_{2n} is the smooth function with $H_2 = 0$, and const. is a constant independent of h and x. By similar considerations, we define the asymptotic expansion of the global error for the j-th neighboring boundary value problem (NP) (4),

$$\zeta_i^{[j]} - P^{[j]}(x_i) = \sum_{n=1}^{m+1} h^{2n}\, E_{2n}^{[j]}(x_i) + \Delta_i^{[j]}, \tag{9}$$

where $\| \Delta_i^{[j]} \| = O(h^{2m+4})$,

$$\zeta_i^{[j]} = \begin{bmatrix} \eta_i^{[j]} \\ \xi_i^{[j]} \end{bmatrix}, \quad P^{[j]}(x) = \begin{bmatrix} p^{[j]}(x) \\ q^{[j]}(x) \end{bmatrix}, \quad E_{2n}^{[j]}(x_i) = \begin{bmatrix} e_{y,2n}^{[j]}(x_i) \\ e_{z,2n}^{[j]}(x_i) \end{bmatrix},$$

and $E_{2n}^{[j]}(x)$ is the smooth solution to the following system of linear boundary value problem.

$$\begin{aligned} E_{2n}^{[j]\prime}(x) &= J\big(P^{[j]}\big)E_{2n}^{[j]}(x) + \ell_{2n+1}P^{[j](2n+1)}(x) \\ &\quad + \text{const.}\, H_{2n}(F_2^{[j]}, F_4^{[j]}, \dots, F_{2n-2}^{[j]}, P^{[j]}, P^{[j]\prime}, \dots, P^{[j](2n-2)}, K_1^{[j]}, K_2^{[j]}) \end{aligned} \tag{10}$$

with the boundary conditions

$$M_0 E_{2n}^{[j]}(0) + M_1 E_{2n}^{[j]}(1) = 0,$$

where

$$J(P^{[j]}) = \begin{bmatrix} g_y(p^{[j]}, q^{[j]}) & g_z(p^{[j]}, q^{[j]}) \\ -f_y(x, p^{[j]}) & 0 \end{bmatrix}$$

and $K_1^{[j]}, K_2^{[j]}$ are obtained by substituting $P^{[j]}(x)$ instead of $Y(x)$ inK_1, K_2.

4. Error Analysis

Since the 2×2 Jacobian matrix $J(Y)$ is the smooth matrix valued function for all $Y \in \mathbb{R}^2$, there exist fundamental matrices $Q(x)$ and $Q^{[j]}(x)$ which satisfy the corresponding homogeneous parts of differential equation systems (7) and (10) respectively,

$$\frac{d}{dx} Q(x) = J(Y) Q(x) \tag{11}$$

$$\frac{d}{dx} Q^{[j]}(x) = J(P^{[j]}) Q^{[j]}(x), \quad \forall x \in [0,1]. \tag{12}$$

Then the solutions to linear non-homogeneous boundary value problems (7) and (10) can be represented in the form

$$E_{2n}(x) = \int_0^1 G\,(x;\xi)(\ell_{2n+1} Y^{2n+1}(\xi) + \text{const.} H_{2n}(\xi)) d\xi \tag{13}$$

$$E_{2n}^{[j]}(x) = \int_0^1 G^{[j]}\,(x;\xi) \left(\ell_{2n+1} P^{[j]2n+1}(\xi) + \text{const.} H_{2n}^{[j]}(\xi)\right) d\xi, \tag{14}$$

where $G(x;\xi)$ and $G^{[j]}(x;\xi)$ are Green's matrices are defined by

$$G(x;\xi) = \begin{cases} Q(x) L^{-1} M_0 Q(0) Q^{-1}(\xi), & \xi < x \\ -Q(x) L^{-1} M_1 Q(1) Q^{-1}(\xi), & \xi > x \end{cases} \tag{15}$$

$$G^{[j]}(x;\xi) = \begin{cases} Q^{[j]}(x) L^{-1} M_0 Q^{[j]}(0) Q^{-1[j]}(\xi), & \xi < x \\ -Q^{[j]}(x) L^{-1} M_1 Q^{[j]}(1) Q^{-1[j]}(\xi), & \xi > x, \end{cases} \tag{16}$$

and L is an invertible matrix such that $L = M_0 Q(0) + M_1 Q(1) = M_0 Q^{[j]}(0) + M_1 Q^{[j]}(1)$. Notice that; in the following statements, $\|\cdot\|$ corresponds to the induced matrix norm of the maximum norm $\|\cdot\|_\infty$ for vectors and const.is a constant independent of x and h.

Lemma 1: Let $Q(x)$and $Q^{[j]}(x)$ be fundamental matrices of (11) and (12) respectively, $G(x;\xi)$ and $G^{[j]}(x;\xi)$ be the Green's functions defined in equations (15) and (16) respectively, then for all $x \in [0,1]$

$$\| Q(x) - Q^{[j]}(x) \| \le \text{const.} \| Y(x) - P^{[j]}(x) \|, \tag{17}$$

$$\begin{gathered} \| Q^{-1}(x) - Q^{-1[j]}(x) \| \le \text{const.} \| Y(x) - P^{[j]}(x) \|, \\ \| G(x;\xi) - G^{[j]}(x;\xi) \| \le \text{const.} \| Y(x) - P^{[j]}(x) \|. \end{gathered} \tag{18}$$

Proof: The subtraction of the corresponding integral equations of (11) and (12), adding, $\pm J(P^{[j]})Q(t)$ and taking the norm of both sides give

$$\| Q(x) - Q^{[j]}(x) \| \leq \| Q(0) - Q^{[j]}(0) \| + \int_0^x \| J(Y) - J(P^{[j]}) \| \| Q(t) \| \, dt$$
$$+ \int_0^x \| J(P^{[j]}) \| \| Q(t) - Q^{[j]}(t) \| \, dt.$$

For all $x \in [0,1]$, it is easily shown that

$$\| J(Y) - J(P) \| \leq \text{const.} \| Y(x) - P(x) |.$$

And by applying the Gronwall's inequality (see in [3]) it is obtained that

$$\| Q(x) - Q^{[j]}(x) \| \leq \text{const.} \| Y - P^{[j]} \|, \; x \in [0, \xi).$$

For equation (18) we can easily deduce that Q^{-1} and $Q^{-1[j]}$ satisfy the following differential equation systems

$$\frac{d}{dx} Q^{-1}(x) = -Q^{-1}(x) J(Y) \quad \text{and} \quad \frac{d}{dx} Q^{-1[j]}(x) = -Q^{-1[j]}(x) J(P^{[j]}).$$

Therefore, an argument similar to the one used in the above statements shows equation (18). For $0 \leq x < \xi$, we have

$$G(x; \xi) = Q(x) L^{-1} M_0 Q(0) Q^{-1}(\xi) \tag{19}$$

$$G^{[j]}(x; \xi) = Q^{[j]}(x) L^{-1} M_0 Q^{[j]}(0) Q^{-1[j]}(\xi). \tag{20}$$

Setting $S = L^{-1} M_0 Q(0) = L^{-1} M_0 Q^{[j]}(0)$ and subtracting (19) from (20) gives

$$G(x; \xi) - G^{[j]}(x; \xi) = Q(x) S Q^{-1}(\xi) - Q^{[j]}(x) S Q^{-1[j]}(\xi).$$

Inserting $\pm Q^{[j]}(x) S Q^{-1}(\xi)$ to the right hand side of the above equation and taking the norm of both sides yield

$$\| G(x; \xi) - G^{[j]}(x; \xi) \| \leq \| Q(x) - Q^{[j]}(x) \| \| S Q^{-1}(\xi) \|$$
$$+ \| Q^{[j]}(x) S \| \| Q^{-1}(\xi) - Q^{-1[j]}(\xi) \|.$$

Since $Q(x)$ and $Q^{-1}(x)$ are continuous on $[0,1]$, applying (17) and (18) follows that

$$\| G(x; \xi) - G^{[j]}(x; \xi) \| \leq \text{const.} \| Y(x) - P^{[j]}(x) \|.$$

Similar arguments hold for $\xi < x \leq 1$. □

Lemma 2: For problems (2), (4) and for all $x \in [0,1]$, we have

$$\| \frac{d^k}{dx^k} (P^{[0]}(x) - Y(x)) \| \leq \begin{cases} \text{const.} h^2 & \text{for} \quad k = 0,1, \dots, 2m \\ \text{const.} h & \text{for} \quad k = 2m+1 \\ \text{const.} & \text{for} \quad k = 2m+2 \end{cases},$$

$$\parallel E_2(x) - E_2^{[0]}(x) \parallel \leq \text{const.} \parallel Y''(x) - P^{[0]''}(x) \parallel, \tag{21}$$

where $P^{[0]}(x) = \begin{bmatrix} p^{[0]}(x) \\ q^{[0]}(x) \end{bmatrix} : [0,1] \to \mathbb{R}^2$ is a vector valued B-spline interpolating polynomial of odd degree $2m+1$ $(m \in \mathbb{Z}^+)$ with a const. not depending on x and h.

Proof: We know that $P^{[0]}(x)$ does not interpolate the exact value of $Y(x)$ at $x = x_i$ so we need to define the auxiliary function

$$\psi^{[0]}(x) = Y(x) + \textstyle\sum_{n=1}^{m+1} h^{2n} E_{2n} + \Delta_i(x), \tag{22}$$

where $\psi^{[0]}(x_i) = \begin{bmatrix} u_i^{[0]} \\ v_i^{[0]} \end{bmatrix}$, $i = 0,1,\dots,N$. Therefore the $2m+1$ odd degree B-spline interpolating polynomial $P^{[0]}(x)$ interpolate $\psi(x)$ at $x = x_i$. From [8] we have

$$\parallel \tfrac{d^k}{dx^k}(P^{[0]}(x) - \psi(x)) \parallel = O(h^{2m+2-k}), \quad \forall x \in [0,1]. \tag{23}$$

Using the k^{th} derivative of the identity $P^{[0]}(x) - Y(x) = P^{[0]}(x) \pm \psi(x) - Y(x)$ and we get

$$\frac{d^k}{dx^k}(P^{[0]}(x) - Y(x)) = \frac{d^k}{dx^k}(P^{[0]}(x) - \psi(x)) + \frac{d^k}{dx^k}(\psi(x) - Y(x)).$$

Hence the equations (22) and (23) gives

$$\parallel \frac{d^k}{dx^k}(P^{[0]}(x) - Y(x)) \parallel = O(h^{2m+2-k}) + O(h^2) \quad \text{as} \quad h \to 0, \ \forall x \in [0,1].$$

Subtracting $E_2^{[0]}(x)$ from $E_2(x)$ in the equations (13) and (14) and inserting $\pm G^{[0]}(x;\xi)Y'''(\xi)$ we get

$$\begin{aligned} E_2(x) - E_2^{[0]}(x) = &\int_0^1 (G(x;\xi) - G^{[0]}(x,\xi))Y'''(\xi)d\xi \\ &+ \textstyle\int_0^1 G^{[0]}(x;\xi)(Y'''(\xi) - P^{[0]'''}(\xi))d\xi. \end{aligned} \tag{24}$$

Now consider the second term of the right hand side of (24) as

$$\begin{aligned} \int_0^1 G^{[0]}(x;\xi)(Y'''(\xi) - P^{[0]'''}(\xi))d\xi = &\int_0^x G^{[0]}(x;\xi)(Y'''(\xi) - P^{[0]'''}(\xi))d\xi \\ + \int_x^1 G^{[0]}(x;\xi)(Y'''(\xi) - P^{[0]'''}(\xi))d\xi \quad &= I_1 + I_2, \end{aligned}$$

where I_1 and I_2 represents the corresponding integrals. Applying integration by parts for I_1, we obtain

$$\begin{aligned} I_1 = &G^{[0]}(x;x^-)(Y''(x) - P^{[0]''}(x)) - G^{[0]}(x;0)(Y''(0) - P^{[0]''}(0)) \\ &- \int_0^x G_\xi^{[0]}(x;\xi)(Y''(\xi) - P^{[0]''}(\xi))d\xi. \end{aligned}$$

Using the derivative properties of spline interpolation, i.e. $Y''(0) = P^{[0]''}(0)$ and substituting the corresponding term of $G(x; x^-)$ we get

$$I_1 = Q(x)L^{-1}M_0Q(0)Q^{-1}(x)(Y''(x) - P^{[0]''}(x))$$
$$-\int_0^x Q(x)L^{-1}M_0Q(0)Q_\xi^{-1}(\xi)(Y''(\xi) - P^{[0]''}(\xi))d\xi.$$

By applying similar arguments for I_2 and using$Y''(1) = P^{[0]''}(1)$, we get

$$I_2 = Q(x)L^{-1}M_1Q(1)Q^{-1}(x)(Y''(x) - P^{[0]''}(x))$$
$$+\int_x^1 Q(x)L^{-1}M_1Q(1)Q_\xi^{-1}(\xi)(Y''(\xi) - P^{[0]''}(\xi))d\xi.$$

Since$Q(x)L^{-1}M_0Q(0)Q^{-1}(x) + Q(x)L^{-1}M_1Q(1)Q^{-1}(x)$ is identity matrix, we derive

$$I_1 + I_2 = (Y''(x) - P^{[0]''}(x)) + \int_x^1 G_1(x;\xi)(Y''(\xi) - P^{[0]''}(\xi))d\xi$$
$$-\int_0^x G_2(x;\xi)(Y''(\xi) - P^{[0]''}(\xi))d\xi,$$

where $G_1(x;\xi) = Q(x)L^{-1}M_1Q(1)Q_\xi^{-1}(\xi)$ and $G_2(x;\xi) = Q(x)L^{-1}M_0Q(0)Q_\xi^{-1}(\xi)$. If we substitute $I_1 + I_2$ in the equation (24) and taking the norm of both sides we get

$$\left\|E_2(x) - E_2^{[0]}(x)\right\| \le \left\|Y''(x) - P^{[0]''}(x)\right\| + \int_0^1 \| G(x;\xi) - G^{[0]}(x,\xi) \| \| Y'''(\xi)\| d\xi$$
$$+\int_x^1 \| G_1(x;\xi) \| \| Y'' - P^{[0]''} \| d\xi - \int_0^x \| G_2(x;\xi) \| \| Y'' - P^{[0]''} \| d\xi. \quad (25)$$

Since $\| G(x;\xi) \|, \| G_1(x;\xi) \|, \| G_2(x;\xi) \|$ are bounded for $0 \le x \le 1$ and $0\le \xi \le 1$, combining the results in Lemma 1 with the equation (25), we obtain the equation (21). □

Lemma 3: Let $E_{2n}(x)$ and $E_{2n}^{[0]}(x)$ be defined in (13) and (14), then for all $x \in [0,1]$

$$\| E_{2n} - E_{2n}^{[0]} \| \le \text{const.} \| Y^{(2n)}(x) - P^{[0](2n)}(x) \|, \quad (26)$$

for $n = 1,2,\dots,m+1$.

Proof: For $n = 1$, it is proved in lemma 2. Suppose that the assumption is true for $n - 1$ with$n \ge 2$. The subtraction $E_{2n}^{[0]}(x)$ from $E_{2n}(x)$ yields

$$E_{2n} - E_{2n}^{[0]} = \ell_{2n+1}\int_0^1 [G(x;\xi)Y^{(2n+1)}(\xi) - G^{[0]}(x;\xi)P^{[0](2n+1)}(\xi)]d\xi$$
$$+\text{const.}\int_0^1 [G(x;\xi)H_{2n}(\xi) - G^{[0]}(x;\xi)H_{2n}^{[0]}(\xi)]d\xi \quad (27)$$

By adding $\pm G^{[0]}(x;\xi)H_{2n}$ to the second part of the above integral and taking the norm of both sides we obtain

$$\int_0^1 \| G(x;\xi)H_{2n} - G^{[0]}(x;\xi)H_{2n}^{[0]} \| \, d\xi \le \text{const.} \| G(x;\xi) - G^{[0]}(x;\xi) \|$$
$$+\text{const.} \| H_{2n} - H_{2n}^{[0]} \|.$$

Hence; from the similar consideration as in the lemma 2 for $x \in [0,1]$, we deduce that

$$\| E_{2n}(x) - E_{2n}^{[0]}(x) \| \le \text{const.} \| Y^{(2n)}(x) - P^{[0](2n)}(x) \| + \text{const.} \| Y(x) - P^{[0]}(x) \|$$
$$+\text{const.} \| H_{2n} - H_{2n}^{[0]} \|. \tag{28}$$

From the induction hypothesis and taking the derivatives of the differential systems (7) and (10), we conclude that

$$\| E_{2i}^{(2n-2i)} - E_{2i}^{[0](2n-2i)} \| \le \text{const.} \| Y^{(2n)}(x) - P^{[0](2n)}(x) \|, \tag{29}$$

$$\| Y^{(2n-i)}(x) - P^{[0](2n-i)}(x) \| \le \text{const.} \| Y^{(2n-2)}(x) - P^{[0](2n-2)}(x) \| \tag{30}$$

for $i = 2, \dots, 2n-1$ and also

$$\| K_1 - K_1^{[0]} \| \le \text{const.} \| Y(x) - P^{[0]}(x) \|, \tag{31}$$

$$\| K_2 - K_2^{[0]} \| \le \text{const.} \| Y(x) - P^{[0]}(x) \|, \tag{32}$$

where K_1 and K_2 depend on sufficiently smooth functions of f and g. Combining the inequalities (29), (30), (31) and (32) we get

$$\| H_{2n} - H_{2n}^{[0]} \| \le \text{const.} \| Y^{(2n)}(x) - P^{[0](2n)}(x) \|. \tag{33}$$

Substituting (33) in the inequality (28) gives (26). □

Lemma 4: For the problems (OP)(2) and (NP)(4), with $j = 0,1,\dots,m-1$, for all $x \in [0,1]$

$$\left\| \frac{d^k}{dx^k}\left(P^{[j]}(x) - Y(x)\right) \right\| \le \begin{cases} \text{const.}\, h^{2j+2} & \text{for} \quad 0 \le k \le 2m-2j \\ \text{const.}\, h^{2m-k+2} & \text{for} \quad 2m-2j+1 \le k \le 2m+2, \end{cases} \tag{34}$$

$$\left\| E_{2n}(x) - E_{2n}^{[j]}(x) \right\| \le \begin{cases} \text{const.}\, h^{2j+2} & \text{for} \quad 1 \le n \le m-j \\ \text{const.}\, h^{2m-2n+2} & \text{for} \quad m-j+1 \le n \le m+1, \end{cases} \tag{35}$$

where $P^{[j]}(x)$ is B-spline polynomial of fixed degree $2m+1$ and $E_{2n}(x)$, $E_{2n}^{[j]}(x)$ satisfy the equations (13) and (14) respectively.

Proof*:* For $j = 0$ the inequalities, (34) and (35) are proved in lemma 2 and 3. Suppose that (34) is true for $1 \le j \le m-2$. For $j = m-1$, we define a new function

$$\psi^{[m-1]}(x) = Y(x) + \sum_{n=1}^{m+1} h^{2n}\left(E_{2n}(x) - E_{2n}^{[m-2]}(x)\right) + \left(\Delta(x) - \Delta^{[m-2]}(x)\right), \tag{36}$$

where $(\Delta(x) - \Delta^{[m-2]}(x))$ is polynomial of degree $2m+1$ which interpolates the values

$$(\Delta_i - \Delta_i^{[m-2]}) = O(h^{2m+4})$$

and $\| \frac{d^k}{dx^k}(\Delta(x) - \Delta^{[m-2]}(x)) \| = O(h^{2m+4-k})$ for $k = 0,1, \dots ,2m + 1$. Taking the k^{th} derivative of (36), we get, for $k = 0,1, \dots ,2m + 1$,

$$\frac{d^k}{dx^k}\left(\psi^{[m-1]}(x) - Y(x)\right) = \sum_{n=1}^{m+1} h^{2n} \frac{d^k}{dx^k}\left(E_{2n}(x) - E_{2n}^{[m-2]}(x)\right) + O\left(h^{2m+4-k}\right), \tag{37}$$

The term $\frac{d^k}{dx^k}(E_{2n}(x) - E_{2n}^{[m-2]}(x))$ depends on $\frac{d^k}{dx^k}(P^{[m-2](2n)}(x) - Y^{(2n)}(x))$. Thus by induction hypothesis we get

$$\left\|\frac{d^{k+2n}}{dx^{k+2n}}(P^{[m-2]}(x) - Y(x))\right\| \le \begin{cases} \text{const.}\, h^{2m-2} & \text{for} \quad 0 \le k \le 4 - 2n \\ \text{const.}\, h^{2m-2n-k+2} & \text{for} \quad 5 - 2n \le k \le 2m - 2n + 2. \end{cases} \tag{38}$$

Substitution (38) in (37) yields

$$\| \frac{d^k}{dx^k}(\psi^{[m-1]}(x) - Y(x)) \| \le \begin{cases} \text{const.}\, h^{2m} & \text{for} \quad 0 \le k \le 2 \\ \text{const.}\, h^{2m-k+2} & \text{for} \quad 3 \le k \le 2m. \end{cases} \tag{39}$$

The identity

$$\frac{d^k}{dx^k}(P^{[m-1]}(x) - Y(x)) = \frac{d^k}{dx^k}(P^{[m-1]}(x) - \psi^{[m-1]}(x)) + \frac{d^k}{dx^k}(\psi^{[m-1]}(x) - Y(x))$$

and from [8] with (39) implies (34). By using similar technique in proof of lemma 3 and by inequality (34) the inequality (35) can be showed. □

Theorem. In case of the algorithms for (OP) (2) and (NP) (4), if we choose the interpolating B-spline polynomials of $2m + 1$ degree($m \in \mathbb{Z}^+$), then for all $x_i \in [0,1]$

$$\| U_i^{[j]} - Y(x_i) \| \le \text{const.}\, h^{2j+2}, \tag{40}$$

for $j = 0,1, \dots , m$ and const.is independent of h and x.

Proof: From the iteration, we write

$$U_i^{[j+1]} = U_i^{[0]} - (\zeta_i^{[j]} - P^{[j]}(x_i)). \tag{41}$$

If we subtract (9) from (6) we obtain

$$\| U_i^{[j+1]} - Y(x_i) \| \le \sum_{n=1}^{m+1} h^{2n} \| E_{2n}(x_i) - E_{2n}^{[j]}(x_i) \| + \| \Delta_i - \Delta_i^{[j]} \|, \tag{42}$$

where $\left\|\Delta_i - \Delta_i^{[j]}\right\| = O(h^{2m+4})$. Then by using the inequalities (35) in the lemma 4, we get

$$\left\|U_i^{[j]} - Y(x_i)\right\| \le \sum_{n=1}^{m-j+1} \text{c onst.}\, h^{2(j-1)+2+2n} + \sum_{n=m-j+2}^{m+1} \text{c onst.}\, h^{2m+2} + \text{const.}\, h^{2m+4}$$
$$\le \text{const.}\, h^{2j+2}. \tag{43}$$

To increase the order of convergence, the relation $2j + 2 \leq 2m + 2$ must hold for $j = 0,1, \dots, j_{max}$ and it implies that $j \leq m$ and $j_{max} = m$. So the results (40) is obtained from (43) for all $x_i \in [0,1]$.

5. Numerical Results

In this section, we use Example 1 in [24] with known solution to verify the theoretical results. In addition, in Example 2 we solve Troesch's problem to exhibit the efficiency of the IDeC method for $\lambda \geq 10$ by comparing the results in [6], [23] .

Example 1.

$$f(x,y) = \frac{\pi^2(\sin\pi x + \sin^4 \pi x + 3\sin^2 \pi x \cos^2 \pi x)}{(1 + \sin^4 \pi x)^2} + \sin^4 \pi x - y^4(x),$$

$$k(y) = \frac{1}{1 + y^3}, \quad \alpha = \beta = 0,$$

with the exact solution $y(x) = \sin \pi x$.

Example 2. The Troesch's problem is defined by

$$y'' = \sinh(\lambda y)$$
$$y(0) = 0, \ y(1) = 1$$

The numerical results for Example 1 is given in Table 1. In the tables, the data's about the IDeC iterates $j = 0,1, \dots, j_{\max}$ are given and $j = 0$ denotes the results of implicit trapezoidal method. To demonstrate the accuracy of the numerical solution$u_i^{[j]}$, we calculate the order of maximum error which is defined by

$$p = \log(e_h/e_{h/2})/\log 2$$

and $2m + 1$ represents the degree of the B-spline polynomials. We use two different step sizes h and $h/2$ respectively and investigate the corresponding errors $e_h, e_{h/2}$ and their observed orders for various IDeC steps. The results of these experiments indicate the increasing order of convergence of IDeC steps and observed orders given in the tables well confirm the theoretical results. In [24], maximum convergence order is $O(h^4)$ for Example 1. However, in our results we obtain $O(h^{2m+2})$. And also, the efficiency of the IDeC method is illustrated for transformed Troesch's problem by $u(x) = \tanh(\lambda y(x)/4)$ which is known as an inheretly unstable two-point boundary value problem. In Table 2, we present the errors of the solution to transformed Troesch's problem with the IDeC steps by comparing the accurate results available in [23] and [6] for $\lambda = 10$. In Table 3, the solutions of transformed Troesch's problem and the last accurate results in [23] for $\lambda = 30, 50$ are given with the same step size h.

It is seen from Table 3 that both results are almost same. However, our method is more effective since [23] uses polynomials of degree 30, 50 respectively, but our results obtained using polynomials degree of 5.

In Table 4, the observed orders are given to emphasize the increasing order of convergence of IDeC steps for $\lambda = 30,50$ using the B-spline polynomials of degree5. The orders are obtained by,

$$p = \log|(y_h - y_{h/2})/(y_{h/2} - y_{h/4})|/\log 2,$$

where $y_h, y_{h/2}, y_{h/4}$ are approximate solutions corresponding to the different stepsizes $h, h/2, h/4$ respectively.

Table 1: Maximum error moduli and observed orders for Example 1

2m+1	*h*	*j=0*	*j=1*	*j=2*	*j=3*
	1/32	*1.398(-03)*	*8.148(-06)*		
3	*1/64*	*3.503(-04)*	*5.710(-07)*		
	1/128	*8.765(-05)*	*3.638(-08)*		
Observed orders		*1.99614*	*3.83488*		
		1.99904	*3.97249*		
	1/32	*1.398(-03)*	*1.682(-05)*	*3.447(-07)*	
5	*1/64*	*3.503(-04)*	*1.032(-06)*	*6.819(-09)*	
	1/128	*8.765(-05)*	*6.482(-08)*	*1.0989(-10)*	
Observed orders		*1.99614*	*4.02675*	*5.65961*	
		1.99904	*4.02675*	*5.95551*	
	1/32	*1.398(-03)*	*1.562(-05)*	*9.401(-07)*	*2.323(-08)*
7	*1/64*	*3.503(-04)*	*1.037(-06)*	*6.726(-09)*	*1.2162(-10)*
	1/128	*8.765(-05)*	*6.489(-08)*	*1.111(-10)*	*4.668(-13)*
		1.99614	*3.91347*	*7.12685*	*7.57803*
		1.99904	*3.99846*	*5.91966*	*8.02538*

Table 2: Errors for Troesch's problem with λ=10

x	*j=0*	*j=1*	*j=2*	*j=3*	Chang[6]	Temimi[23]
0.1	*3.041(-07)*	*2.688(-10)*	*6.281(-11)*	*6.247(-11)*	*5.821(-11)*	*6.248(-11)*
0.2	*8.566(-07)*	*4.846(-10)*	*1.928(-10)*	*1.919(-10)*	*1.794(-10)*	*1.919(-10)*
0.3	*2.086(-06)*	*3.349(-10)*	*5.170(-10)*	*5.158(-10)*	*4.854(-10)*	*5.157(-10)*
0.4	*4.881(-06)*	*1.353(-09)*	*1.349(-09)*	*1.349(-09)*	*1.281(-09)*	*1.349(-09)*
0.5	*1.106(-05)*	*7.924(-09)*	*3.303(-09)*	*3.312(-09)*	*3.171(-09)*	*3.312(-09)*
0.6	*2.409(-05)*	*2.661(-08)*	*6.472(-09)*	*6.517(-09)*	*6.236(-09)*	*6.517(-09)*
0.7	*4.924(-05)*	*6.448(-08)*	*2.685(-09)*	*2.839(-09)*	*2.312(-09)*	*2.833(-09)*
0.8	*9.068(-05)*	*1.511(-07)*	*6.585(-10)*	*2.373(-10)*	*1.121(-09)*	*2.356(-10)*
0.9	*1.394(-04)*	*2.288(-07)*	*3.279(-09)*	*2.389(-09)*	*3.567(-09)*	*2.386(-09)*

Table 3: Solutions for Troesch's problem with λ=30, 50

	λ=30		λ=50	
x	*j=2*	Temimi[23]	*j=2*	Temimi[23]
0.1	*2.499825044(-13)*	*2.498427550(-13)*	*2.289910897(-21)*	*2.168089718(-21)*
0.2	*5.033478719(-12)*	*5.031718066(-12)*	*3.398683397(-19)*	*3.269919297(-19)*
0.3	*1.011007423(-10)*	*1.010808107(-10)*	*5.044093407(-17)*	*4.917006047(-17)*
0.4	*2.030662723(-09)*	*2.030470452(-09)*	*7.486098374(-15)*	*7.372216233(-15)*
0.5	*4.078695111((-08)*	*4.078557034(-08)*	*1.111035509(-12)*	*1.102228564(-12)*
0.6	*8.192278125(-07)*	*8.192233710(-07)*	*1.648922897(-10)*	*1.643649842(-10)*
0.7	*1.645463056(-05)*	*1.645463961(-05)*	*2.447218563(-08)*	*2.445464917(-08)*
0.8	*3.305007649(-04)*	*3.304990272(-04)*	*3.631994383(-06)*	*3.632153512(-06)*
0.9	*6.643764763(-03)*	*6.643689371(-03)*	*5.390439175(-04)*	*5.389856648(-04)*
0.95	*3.025969663(-02)*	*3.02593407(-02)*	*6.581608721(-03)*	*6.580132361(-03)*
0.97	*5.753258144(-02)*	*5.75318891(-02)*	*1.815582947(-02)*	*1.815179410(-02)*
0.98	*8.222382385(-02)*	*8.22231682(-02)*	*3.087747331(-02)*	*3.087419365(-02)*
0.99	*1.269719232(-01))*	*1.26969423(-02)*	*5.627316454(-02)*	*5.625810248(-02)*

Table 4: The orders for λ=30, 50 with degree of 5

λ	*j=0*	*j=1*	*j=2*
30	*1.94194*	*3.95678*	*5.54841*
40	*1.85135*	*3.86983*	*5.68889*
50	*1.69808*	*3.71666*	*5.75247*

Conclusion

We give a numerical treatment for a class of nonlinear boundary value problems by iterated defect correction method (IDeC) based on the implicit trapezoid method using B-spline piecewise polynomials. We don't need to solve the piecewise neighboring problem since the derivative properties and the advantage of the construction of B-spline polynomials. The maximum attainable order of the defect correction steps that depend on the degree of the polynomial are given in the theorem. We observed that the orders in the given tables show good agreement with the order sequence to be expected from theory. And we also overcome the difficulty in solving the Troesch's problem for large values of λ by increasing the order of convergence. It is expected that this approach can be used to the other unstable or strongly nonlinear two point boundary value problems.

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