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# Equivalence frames and their duals in Hilbert spaces

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## Abstract

Theory of frames has very applications to various areas of science and engineering. The paper presents all duals of a constructed frame that is obtained by an operator and a primary frame. By two frames and their frame operators, a frame is given that has an identify frame operator. Moreover, a relation between ordinary duals and operator duals is given. Finally, an equation is introduced for pre-frame operator of a given frame and pre-frame operator of its operator dual.

**Keywordsandphrases:** Dualframe, Equivalent frames, Frameoperator, frame, Operator dualframe.

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## 1. Introduction

Since 1950, frames havebeen a useful and important tool in signal processing, image processing, data compression, sampling theory, etc. In the frame theory, dual frames and operators corresponding to a frame play an important role. A collection of frames corresponding to a given frame is defined which has a special relation with respect to basicframe. They are called dual frames.

Recently, dual frames havebeen extended to generalize dual frames or operator dual frames [2]. In the paper, we consider some properties about a constructed frame and operator dual frames by corresponding operators to a given frame.The paper is organized as follows. In reminder of the section some elementary definitions and properties of frames in Hilbert spaces are recalled. Main results section considers the relation between operator dual sofa frame and a generated frame by an invertible and adjoin table operator. Also a frame is constructed with the special frame operator. The relations between dual frames and its operator duals are presented.

Throughout the paper, we fix the notations $J$ for a finite or countablyinfinite index set, and $H$ is a finitely or countably generated Hilbert space.In the reminder of section, we recall the definition and some basic properties of frames. We would like to comment here for more details see [1] and the references therein. The notion of frames for Hilbert spaces has been introduced in [3] as a countable family $\{f_j\}_{j\in J}$ in a Hilbert space $\boldsymbol{H}$ satisfying

$$A\|f\|^2\leq\sum_{j\in J}|\langle f,f_j\rangle|^2\leq B\|f\|^2,$$

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for all $f \in H$ and constants $0 < A \leq B < \infty$, independent of *f*.

Let $\{f_j\}_{j\in J}$ be a frame for H with lower and upper frame bounds A and B, respectively. The pre-frame operator $T : H \longrightarrow l_2$ defined by $T(f) = \{\langle f, f_j\rangle\}_{j\in J}$ which is an injective andclosed range bounded operator and $\| T \| \leq \sqrt{B}$. Also, the frame operator $S : H \longrightarrow H$ is defined by $Sf = \sum_{j\in J} \langle f, f_j\rangle f_j$ thatis positive, invertible and bounded. The inequality $A \leq S \leq B$ holds and the reconstruction satisfies for all $f \in H$ such that $f = \sum_{j\in J} \langle f, S^{-1} f_j\rangle f_j$.

Every element of a Hilbert space has decomposition with respect to a given frame. Dual frames are important in this decomposition,because coefficients are given from a dual frame. In[2], the authors have extended dual frames to generalized dual frames in Hilbert spaces.

Let $\{f_j\}_{j\in J}$ be a frame for H. If there exists a frame $\{g_j\}_{j\in J}$ for H such that

$$f = \sum_{j\in J} \langle f, g_j\rangle f_j, \quad \forall f \in H,$$

then frame $\{g_j\}_{j\in J}$ is called a dual frame of $\{f_j\}_{j\in J}$. Also, assume that $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be two frames for H. If there exists an invertible and bounded operator $\Gamma$ on H such that

$$f = \sum_{j\in J} \langle \Gamma f, g_j\rangle f_j, \quad \forall f \in H,$$

then $(\{g_j\}_{j\in J}, \Gamma)$ is called to be an operator dual of $\{f_j\}_{j\in J}$.

## Main results

It is well known that for every frame, a family of frames is constructed by invertible and bounded operators on H. Now, the set of duals of these frames is characterized with respect to the primary frame.

**Theorem 2.1.**[1] Let $\{f_j\}_{j\in J}$ be a frame for H with frame operator S and lower and upper frame bounds A and B, respectively. Then $\theta \in B(H)$ is surjective if and only if $\{\theta f_j\}_{j\in J}$ is a frame for H. In this case, $S_\theta := \theta S \theta^*$ is frame operator for $\{\theta f_j\}_{j\in J}$.

**Theorem 2.2.**Let $\{f_j\}_{j\in J}$ be a frame forH and let $\theta$ be a bounded invertible operator on H. Then the set of dual of frames $\{\theta f_j\}_{j\in J}$ is

$$\{\{(\theta^{-1})^* g_j\}_{j\in J} \; : \{g_j\}_{j\in J} \textit{ is a dual for } \{f_j\}_{j\in J}\}.$$

**Proof**. First, suppose that $\{g_j\}_{j\in J}$ is a dual of $\{f_j\}_{j\in J}$. For $f \in H$, we obtain

$$\sum_{j\in J} \langle f, (\theta^{-1})^* g_j \rangle \theta f_j = \theta(\sum_{j\in J} \langle \theta^{-1} f, g_j \rangle f_j) = \theta(\theta^{-1}) f = f.$$

Then $\{(\theta^{-1})^* g_j\}$ is a dual for $\{\theta f_j\}_{j\in J}$. Now, Suppose that $\{h_j\}_{j\in J}$ is a dual frame for $\{\theta f_j\}_{j\in J}$. Set $g_j = \theta^* h_j$, for $j \in J$, so for $f \in H$,

$$\sum_{j\in J}\langle f, g_j\rangle f_j = \sum_{j\in J}\langle f, \theta^* h_j\rangle \theta^{-1}\theta f_j = \theta^{-1}(\sum_{j\in J}\langle \theta f, h_j\rangle \theta f_j) = \theta^{-1}\theta f = f$$

Then $\{g_j\}_{j\in J}$ is a dual for $\{f_j\}_{j\in J}$ and $h_j = (\theta^{-1})^* g_j$. □

In general case we can characterize all of operator duals of a constructed frame. This characterization is given in the following theorem.

**Theorem2.3.**Let $\{f_j\}_{j\in J}$ be a frame forH and let $\theta$ be a bounded invertible operator on H. Then the set $\{(\{g_j\}_{j\in J}, \Gamma\theta^{-1}) : (\{g_j\}_{j\in J}, \Gamma) \textit{ is an operator dual for } \{f_j\}_{j\in J}\}$, Contains all of operator duals of $\{\theta f_j\}_{j\in J}$.

**Proof.**Let $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$. Then for $f \in H$,

$$\sum_{j\in J}\langle \Gamma\theta^{-1} f, g_j\rangle \theta f_j = \theta(\sum_{j\in J}\langle \Gamma\theta^{-1} f, g_j\rangle f_j) = \theta(\theta^{-1}) f = f$$

It shows that $(\{g_j\}_{j\in J}, \Gamma\theta^{-1})$ is an operator dual for $\{\theta f_j\}_{j\in J}$.Now, if $(\{g_j\}_{j\in J}, \gamma)$ is an operator dual for $\{\theta f_j\}_{j\in J}$, then it is enough thatwe set $\Gamma := \gamma\theta$, we obtain $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$. □

The following theorem constructs a frame with the frame operator that is a combination of two frame operators.

**Theorem 2.4**. If $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are frames with frame operators $S_F$ and $S_G$, respectively, then there exists an bounded and invertible operator $\theta$ on H such that $S_F$ is the frame operator of $\{\theta g_j\}_{j\in J}$.

Proof.For the bounded and invertible operator $\theta = S_F^{\frac{1}{2}} S_G^{-\frac{1}{2}}$, the sequence $\{\theta g_j\}_{j\in J}$ is a frame with frame operator $S_\theta = \theta S_G \theta^*$,by Theorem 2.1. So $S_\theta := \theta S_G \theta^* = (S_G^{\frac{1}{2}} S_G^{-\frac{1}{2}}) S_G (S_F^{\frac{1}{2}} S_G^{-\frac{1}{2}})^* = S_F$. □

Two sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are said to be equivalent sequences if there exists a bounded invertible operator on H such that for $j \in J, \Lambda f_j = g_j$.

In[2], it is shown that every dual frame is an operator dual, but we nowsee another relation between dual frames and operator dual frames of a frame.

**Theorem 2.5.**Let $\{f_j\}_{j\in J}$ be a frame for H, and let $\{g_j\}_{j\in J}$ be a sequence in H.Then $\{g_j\}_{j\in J}$ is equivalent to a dual frame of $\{f_j\}_{j\in J}$ if and only if there is a bounded invertible operator $\Lambda$ such that $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$.

Proof.For the proof of "if" part, assume that there is a dual frame $\{h_j\}_{j\in J}$ for $\{f_j\}_{j\in J}$ such that $\{h_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are equivalent. Therefore, there is a bounded invertible operator $\Lambda : H \longrightarrow H$, with $\Lambda g_j = h_j$, for all $j \in J$. By Theorem 2.1, the sequence $\{g_j\}_{j\in J}$ is a frame.

On the other hand, for $f \in H$, $f = \sum_{j\in J} \langle f, h_j\rangle f_j = \sum_{j\in J} \langle f, \Lambda g_j\rangle f_j$, then $f = \sum_{j\in J} \langle \Lambda^* f, g_j\rangle f_j$. It shows that $(\{g_j\}_{j\in J}, \Lambda^*)$ is an operator dual for $\{f_j\}_{j\in J}$.
The proof of converse is clear.□
Moreover, the equivalent frames have the same Grammian matrices.

**Proposition 2.6.**Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be equivalent Parseval frames and let $G_F$ and $G_G$ are Grammian matrices of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ respectively. Then $G_F = G_G$.

Proof. Two frames $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are equivalent, so there exists a bounded invertible operator $\theta : H \longrightarrow H$ with $\theta f_j = g_j$ for $j \in J$. Since the frame operators of these frames areidentity operator on H, and by Theorem 2.1,

$$\theta id\, \theta^* = id \Rightarrow \theta\theta^* = id.$$

So $\theta$ is a unitary operator and then for $i, j \in J$,
$\langle g_i, g_j\rangle = \langle \theta f_i, \theta f_j\rangle = \langle f_i, f_j\rangle,$

it shows that $G_F = G_G$.□
The relation between pre-frame operators of a Parseval frame and its operator duals is obtained in the following theorem.

**Theorem 2.7**.Let $\{f_j\}_{j\in J}$ be a Parseval frame for H with pre-frame operator $T_F$ and let $\{g_j\}_{j\in J}$ be a frame with pre-frame operator $T_G$. Then $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$ if and only if $P_{T_F} T_G \Gamma = T_F$, where $P_{T_F}$ is the orthogonal projection on the range of $T_F$.

Proof.Suppose that $(\{g_j\}_{j\in J}, \Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$. The pre-frame operator $T_F$isisometry and $\{f_j\}_{j\in J}$ is a Parseval frame. For $f \in H$,

$$\langle P_{T_F} \Gamma T_G f, T_F f\rangle = \langle T_G \Gamma f, P_{T_F} T_F f\rangle = \langle T_G \Gamma f, T_F f\rangle = \langle \sum_{j\in J} \langle \Gamma f, g_j\rangle f_j, f\rangle = \langle f, f\rangle = \langle T_F f, T_F f\rangle,$$

Then $P_{T_F} T_G \Gamma = T_F$. Conversely, since $T_F$ is isometry, similar to the above equalities we have

$$\langle f, g\rangle = \langle T_F f, T_F g\rangle = \langle P_{T_F} T_G \Gamma f, T_F g\rangle = \langle \sum_{j\in J} \Gamma f, g_j\rangle f_j, g\rangle, \quad \forall f \in H,$$

Then $f = \sum_{j\in J} \langle \Gamma f, g_j\rangle f_j, \quad \forall f \in H,$ and $(\{g_j\}_{j\in J}, \Gamma)$ is a dual for $\{f_j\}_{j\in J}$.□

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