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Convergence and Data Dependence Result for Picard S-Iterative Scheme Using Contractive-Like operators

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Abstract

The purpose of this paper is to prove the strong convergence theorem and data dependence result for special types of iterative that is Picard S-iteration dealing with contractive-like operators.

Keyword: Picard S-iterative scheme, Contractive-Like Operators, Data Dependence

1. Introduction and Preliminaries

Let X be a Banach space, $K \subset X$ be a non empty subset and $T: K \to K$ be a map, Gursoy and Karakaya in [2] introduced a Picard S-iteration as

$$x_0 \in K$$
$$x_{n+1} = Ty_n$$
$$y_n = (1-\alpha_n)Tx_n + \alpha_n Tz_n$$

$z_n = (1-\beta_n)x_n + \beta_n Tx_n \qquad n = 0,1,2, \qquad \cdots \quad (1.1)$

Where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying certain control condition.

An operator T is called **contractive like operator** [1] if there exist a constant $q \in (0,1)$ and strictly increasing and continuous function $\varphi: [0,\infty] \to [0,\infty]$ with $\varphi(0) = 0$ such that for each $x, y \in X$

$\|Tx - Ty\| \leq q\|x - y\| + \varphi(\|x - Tx\|) \qquad (1.2)$

In [4] Soltoz and Grosan studied a data dependence result of Ishikawa iterative Scheme using Contractive-Like operator, while Asduzaman and Zulfikar Ali [3] established a data dependence result of Noor iterative Scheme dealing with Contractive-Like operator. In this paper we study data dependence of Picard S-iteration for the same type of operators.

By the same argument of proof lemma (2.4) in [3], we can prove:

Lemma 1.1

Let $\{x_n\}$ be a non-negative sequence for which one supposes there exists $n_0 \in \mathbb{N}$, and a constant k such that for all $n \geq n_0$ one has satisfied the following inequality

$$x_{n+1} \leq k[(1-\delta_n)x_n + \delta_n \sigma_n]$$

Where $\delta_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \delta_n = \infty$, and $\sigma_n \geq 0 \;\; \forall n \in \mathbb{N}$. Then

$$0 \leq \limsup_{n\to\infty} x_n \leq k \limsup_{n\to\infty} \sigma_n$$

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Remark 1.2 [3]

Let $\{d_n\}$ be a non-negative sequence such that $d_n \in (0,1]$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} d_n = \infty$ then $\prod_{n=1}^{\infty}(1 - dn=0$.

$1. Main Results:

In this section we give our main theorems. First we will prove that a sequence $\{x_n\}$ obtain from (1.1) converges uniquely to a fixed point of operator (1.2)

Theorem 2.1:

Let X be a Banach space, K be a non empty closed convex subset of X, and let $T: K \to K$ be a contractive-like operator with fixed point p. Then for all $x_0 \in K$ the Picard S-iteration converges to the unique fixed point of T if $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$.

Proof:

First we prove that the Picard S-iteration converge to the fixed point p of T for all $x_0 \in K$.

From (1.1) and (1.2) we get

$$\|x_{n+1} - p\| = \|Ty_n - Tp\|$$

$$\begin{aligned}
&\le q\|y_n - p\| \\
&\le q\|(1-\alpha_n)Tx_n + \alpha_n Tz_n - p\| \\
&\le q^2(1-\alpha_n)\|x_n - p\| + q^2\alpha_n\|z_n - p\| \\
&\le q^2(1-\alpha_n)\|x_n - p\| + q^2\alpha_n[(1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\|] \\
&\le q^2(1-\alpha_n)\|x_n - p\| + q^2\alpha_n(1-\beta_n)\|x_n - p\| + q^3\alpha_n\beta_n\|x_n - p\| \\
&\le q^2[(1-\alpha_n\beta_n) + q\alpha_n\beta_n]\|x_n - p\|
\end{aligned}$$

Thus by induction we get:

$$\|x_{n+1} - p\| \le q^{2n}\|x_0 - p\|[(1-\alpha_n\beta_n) + q\alpha_n\beta_n][(1-\alpha_{n-1}\beta_{n-1}) + q\alpha_{n-1}\beta_{n-1}]\cdots[(1-\alpha_0\beta_0) + q\alpha_0\beta_0]$$

But $(1-\alpha_n\beta_n) + q\alpha_n\beta_n = 1 - \alpha_n\beta_n(1-q) \qquad \forall n \in \mathbb{N}$

Therefore

$$\|x_{n+1} - p\| \le q^{2n}\prod_{k=0}^{n}\big(1 - \alpha_k\beta_k(1-q)\big)\|x_0 - p\| \qquad \cdots (2.1)$$

Since $\sum_{k=0}^{\infty} \alpha_k\beta_k = \infty$ implies $\sum_{k=0}^{\infty}\big(1 - \alpha_k\beta_k(1-q)\big) = \infty$ for all $q \in (0,1)$. So, using Remark (1.2) we get

$$\lim_{n\to\infty}\prod_{k=0}^{n}\big(1 - \alpha_k\beta_k(1-q)\big) = 0 \qquad \cdots (2.2)$$

From (2.1) and (2.2) we get $\lim_{n\to\infty}\|x_{n+1} - p\| = 0$.

Now to show that the fixed point p is unique, suppose that T has two fixed point p and r then

$$\|p - r\| = \|Tp - Tr\| \le q\|p - r\| + \varphi(\|p - Tp\|) = q\|p - r\|$$

This implies that $p = r$ since $q \in (0,1)$.

Now we discus the data dependence result

Theorem 2.2:

Let X be a real Banach space, $K \subseteq X$ be a nonempty closed convex set, $T: K \to K$ be a contractive-like operator with a fixed point p and $S: K \to K$ be an approximate operator to T with a fixed point p^*, that is, $\|Tz -$

$Sz\| \leq \epsilon$ for all $z \in X$ where ϵ is a fixed number. If $\{x_n\}$ is a sequence generated by (1.1) such that $\beta_n \geq \frac{1}{2}$ and $\alpha_n\beta_n \geq \frac{1}{4}$ for all $n \in \mathbb{N}$, then

$\|p - p^*\| \leq \frac{4q+4q^2+1}{q(1-q)}\epsilon.$

Proof:

For a given x_0, u_0 in K and for all $n \in \mathbb{N}$, the Picard S-iteration for T and S are:

$$x_{n+1} = Ty_n \qquad u_{n+1} = Sv_n$$
$$y_n = (1-\alpha_n)Tx_n + \alpha_n Tz_n \qquad v_n = (1-\alpha_n)Su_n + \alpha_n Sw_n$$
$$z_n = (1-\beta_n)x_n + \beta_n Tx_n \qquad w_n = (1-\beta_n)u_n + \beta_n Su_n$$

Now

$$\|x_{n+1} - u_{n+1}\| = \|Ty_n - Sv_n\| \leq \|Sv_n - Tv_n\| + \|Tv_n - Ty_n\|$$
$$\leq \epsilon + q\|y_n - v_n\| + \varphi(\|y_n - Ty_n\|) \qquad \cdots (2.3)$$

Note that

$$\|y_n - v_n\| = \|(1-\alpha_n)Tx_n + \alpha_n Tz_n - (1-\alpha_n)Su_n - \alpha_n Sw_n\|$$
$$\leq (1-\alpha_n)\|Tx_n - Su_n\| + \alpha_n\|Tz_n - Sw_n\|$$

$$\leq (1-\alpha_n)[\|Su_n - Tu_n\| + \|Tu_n - Tx_n\|] + \alpha_n[\|Sw_n - Tw_n\| + \|Tw_n - Tz_n\|]$$

$$\leq (1-\alpha_n)[\epsilon + q\|u_n - x_n\| + \varphi(\|x_n - Tx_n\|)] + \alpha_n[\epsilon + q\|z_n - w_n\| +$$
$\varphi zn{-}Tzn$ $\cdots 2.4$

But

$$\|z_n - w_n\| = \|(1-\beta_n)x_n + \beta_n Tx_n - (1-\beta_n)u_n - \beta_n Su_n\|$$
$$\leq (1-\beta_n)\|x_n - u_n\| + \beta_n\|Tx_n - Su_n\|$$
$$\leq (1-\beta_n)\|x_n - u_n\| + \beta_n[\|Su_n - Tu_n\| + \|Tu_n - Tx_n\|]$$
$$\leq (1-\beta_n)\|x_n - u_n\| + \beta_n[\epsilon + q\|u_n - x_n\| + \varphi(\|x_n - Tx_n\|)]$$
$$\leq \|x_n - u_n\|[(1-\beta_n) + \beta_n q] + \beta_n\epsilon + \beta_n\varphi(\|x_n - Tx_n\|) \qquad \cdots (2.5)$$

Putting equations (2.3), (2.4), and (2.5) together we get

$$\|x_{n+1} - u_{n+1}\| \leq \epsilon + q\Big[(1-\alpha_n)\big(\epsilon + q\|u_n - x_n\| + \varphi(\|x_n - Tx_n\|)\big) + \alpha_n\Big(\epsilon + \varphi(z_n - Tz_n) +$$
$qxn{-}un1{-}\beta n{+}\beta nq{+}\beta n\epsilon{+}\beta n\varphi xn{-}Txn{+}\varphi yn{-}Tyn$

$$\leq q^2\|x_n - u_n\|[1 - \alpha_n\beta_n(1-q)] + \epsilon(1 + q + q^2\alpha_n\beta_n) + \varphi(\|x_n - Tx_n\|)[q(1-\alpha_n) + q^2\alpha_n\beta_n]$$
$$+ \alpha_n q\varphi(\|z_n - Tz_n\|) + \varphi(\|y_n - Ty_n\|)$$

Since, $q \neq 0$

$$\|x_{n+1} - u_{n+1}\| \leq q^2\Bigg[1 - \alpha_n\beta_n(1-q)\|x_n - u_n\| + \left(\frac{1}{q^2} + \frac{1}{q} + \alpha_n\beta_n\right)\epsilon + \left(\frac{1}{q} - \frac{\alpha_n}{q} + \alpha_n\beta_n\right)\varphi(\|x_n - Tx_n\|)$$
$$+ \frac{\alpha_n}{q}\varphi(\|z_n - Tz_n\|) + \frac{1}{q^2} + \varphi(\|y_n - Ty_n\|)\Bigg]$$

But, $\beta_n \geq \frac{1}{2}$ and $\alpha_n\beta_n \geq \frac{1}{4}$ $\forall n$, thererfore

$$\|x_{n+1} - u_{n+1}\| \leq$$
$$q^2\left[1 - \alpha_n\beta_n(1-q)\|x_n - u_n\| + \alpha_n\beta_n(1-q)\frac{\left(\frac{4}{q^2} + \frac{4}{q} + 1\right)\varepsilon + \left(\frac{2}{q} + 1\right)\varphi(\|x_n - Tx_n\|) + \frac{2}{q}\varphi(\|z_n - Tz_n\|) + \frac{4}{q^2}\varphi(\|y_n - Ty_n\|)}{(1-q)}\right]$$

Since φ is continuous function and all the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge to p the fixed point of T then $\lim_{n\to\infty} \varphi(\|x_n - Tx_n\|) = \lim_{n\to\infty} \varphi(\|y_n - Ty_n\|) = \lim_{n\to\infty} \varphi(\|z_n - Tz_n\|) = 0$.
Now, if we put

$$\delta_n = \alpha_n \beta_n (1-q)$$

$$\sigma_n = \frac{\left(\frac{4}{q^2} + \frac{4}{q} + 1\right)\varepsilon + \left(\frac{2}{q} + 1\right)\varphi(\|x_n - Tx_n\|) + \frac{2}{q}\varphi(\|z_n - Tz_n\|) + \frac{4}{q^2}\varphi(\|y_n - Ty_n\|)}{(1-q)}$$

Then, using Lemma (1.1) we get

$$\lim_{n\to\infty} sup\|x_{n+1} - u_{n+1}\|$$
$$\leq \lim_{n\to\infty} \sup q^2 \frac{\left(\frac{4}{q^2} + \frac{4}{q} + 1\right)\varepsilon + \left(\frac{2}{q} + 1\right)\varphi(\|x_n - Tx_n\|) + \frac{2}{q}\varphi(\|z_n - Tz_n\|) + \frac{4}{q^2}\varphi(\|y_n - Ty_n\|)}{(1-q)}$$

Which implies

$$\|p - p^*\| \leq \frac{4q + 4q^2 + 1}{q(1-q)}\epsilon$$

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