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Some Notes on Lifts of Almost Paracontact Structures

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Abstract

In this paper, we shall study some tensor fields in tangent bundles defined by lifts of paracontact structures.

Keywords: Almost paracontact structure; Nijenhius tensor; ϕ_φ-operator; pure Riemannian metric

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1. Introduction

Let M_n be a paracompact differentiable manifold of dimension n. We denote by $\Im_q^p(M_n)$ the set of all differentiable tensor fields of type (p,q) on M_n, and let $\varphi \in \Im_1^1(M_n)$, $\xi \in \Im_0^1(M_n)$ and $\eta \in \Im_1^0(M_n)$ be a tensor field of type $(1,1)$, a vector field and 1-form on M_n, respectively. If φ, ξ and η satisfy the conditions
$\eta(\xi) = 1,$

(1.1) $$\varphi^2 X = X - \eta(X)\xi$$

for any $X \in \Im_0^1(M_n)$, then M_n is said to have an almost paracontact structure (φ, ξ, η) and M_n is called an almost paracontact manifold ([3], [5]). Then the equations
$\varphi\xi = 0,$

(1.2) $$\eta(\varphi X) = 0,$$

$rank\,\varphi = n - 1$
hold good.

Every almost paracontact manifold M_n admits an associated Riemannian metric tensor field g such that ([2], [5])
$g(X,\xi) = \eta(X),$

(1.3) $$g(X,Y) - \eta(X)\eta(Y) = g(\varphi X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_n)$. The structure (φ, ξ, η, g) is called almost paracontact Riemannian structure on M_n. Let $T(M_n)$ be a tangent bundle of M_n with a natural projection $\pi : T(M_n) \to M_n$ $(\pi : (x^i, y^i) \to (x^i))$, and let f be a function on M_n. Then the vertical lift of f, denoted by f^v is defined by $f^v = f \circ \pi$.

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Let X be a vector field on M_n. The vertical lift of X to $T(M_n)$ denoted by X^v is defined by $X^v(\imath\eta) = (\eta(X))^v$, η being an arbitrary 1-form on M_n and $\imath\eta$ is regarded as a function on $T(M_n)$ in the form: $\imath\eta = y^s\eta_s$.

Then the formulas [3]

$$X^v f^v = 0,\ (fX)^v = f^v X^v,\ I^v X^v = 0, \eta^v(X^v) = 0,$$

(1.4) $$(f\eta)^v = f^v\eta^v, [X^v, Y^v] = 0, \varphi^v X^v = 0,$$

hold good, where $f \in \Im_0^0(M_n),\ X, Y \in \Im_0^1(M_n),\quad \eta \in \Im_1^0(M_n),\ \varphi \in \Im_1^1(M_n),\ I = id_{M_n}$.

The complete lift of the function f on M_n to $T(M_n)$, denoted by f^c, is defined by $f^c = \imath(df)$. The complete lift of X on M_n to $T(M_n)$, denoted by X^c, is defined by $X^c f^c = (Xf)^c$, and the complete lift of η on M_n to $T(M_n)$, denoted by η^c, is defined by $\eta^c(X^c) = (\eta(X))^c$. Then we known that ([3], [5]):

(1.5) $$\begin{aligned} &(fX)^c = f^c X^v + f^v X^c = (Xf)^c, \\ &X^c f^v = (Xf)^v, \eta^v(x^c) = (\eta(x))^v, \\ &X^v f^c = (Xf)^v, \varphi^v X^c = (\varphi X)^v, \\ &\varphi^c X^v = (\varphi X)^v, (\varphi X)^c = \varphi^c X^c, \\ &\eta^v(X^c) = (\eta(X))^c, \eta^c(X^v) = (\eta(X))^v, \\ &I^c = I, I^v X^c = X^v, [X^c, Y^c] = [X, Y]^c, [X^v, Y^c] = [X, Y]^v. \end{aligned}$$

2. Lifts of Almost Paracontact Structures

Let (φ, ξ, η) be a paracontact structure on M_n. From (1,.1) and (1.2), we get on taking complete, vertical and horizontal lifts ([3])

(2.1) $$\begin{aligned} &(\varphi^c)^2 = I - \eta^v \otimes \xi^c - \eta^c \otimes \xi^v, \\ &\varphi^c\xi^v = 0, \varphi^c\xi^c = 0,\ \eta^v \circ \varphi^c = 0, \\ &\eta^c \circ \varphi^c = 0, \eta^v(\xi^v) = 0, \eta^v(\xi^c) = 1, \\ &\eta^c(\xi^v) = 1, \eta^c(\xi^c) = 0 \end{aligned}$$

and

(2.2) $$\begin{aligned} &(\varphi^H)^2 = I - \eta^v \otimes \xi^H - \eta^H \otimes \xi^v, \\ &\varphi^H\xi^v = 0, \varphi^H\xi^H = 0, \eta^v \circ \xi^H = 0, \\ &\eta^H \circ \varphi^H = 0, \eta^v(\xi^v) = 0, \eta^v(\xi^H) = 1, \\ &\eta^H(\xi^v) = 1, \eta^H(\xi^H) = 0. \end{aligned}$$

We now define a (1,1) tensor field $J = \varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c$ on $T(M_n)$. We have

$$\begin{aligned} J^2 X^c &= J(JX^c) = J((\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)X^c) \\ &= J((\varphi X)^c - \eta^v(X^c)\xi^v - \eta^c(X^c)\xi^c) \\ &= J((\varphi X)^c - (\eta(X))^v \xi^v - (\eta(X))^c \xi^c) \end{aligned}$$

$$= (\varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c)\left((\varphi X)^c - (\eta(X))^v \xi^v - (\eta(X))^c \xi^c\right)$$

$$= \varphi^c(\varphi X^c) - \varphi^c((\eta(X))^v \xi^v) - \varphi^c((\eta(X))^c \xi^c) - \xi^v \otimes \eta^v (\varphi(X))^c$$
$$+ \xi^v \otimes \eta^v (\eta(X))^v \xi^v) + \xi^v \otimes \eta^v (\eta(X))^c \xi^c) - \xi^c \otimes \eta^c (\varphi(X))^c$$
$$+ \xi^c \otimes \eta^c ((\eta(X))^v \xi^v + \xi^c \otimes \eta^c ((\eta(X))^c \xi^c$$
$$= (\varphi^c)^2 X^c - (\eta(X))^v \underline{\varphi^c \xi^v} - (\eta(X))^c \underline{\varphi^c \xi^c} - \xi^v (\eta^v (\varphi(X))^c)$$
$$+ (\eta(X))^v (\xi^v \underline{(\eta^v(\xi^v)}) + \xi^v \otimes \eta^v (\eta(X))^c \xi^c) - \xi^c \otimes \eta^c (\varphi(X))^c$$
$$+ \xi^c \otimes \eta^c ((\eta(X))^v \xi^v + \xi^c \otimes \eta^c ((\eta(X))^c \xi^c$$
$$= X^c - (\xi^v \otimes \eta^c) X^c - (\xi^c \otimes \eta^v) X^c - (\eta(X))^v \underline{\varphi^c \xi^v} - (\eta(X))^c \underline{\varphi^c \xi^c}$$
$$- \xi^v (\eta^v (\varphi(X))^c) + (\eta(X))^v (\xi^v \underline{(\eta^v(\xi^v)}) + \xi^v \otimes \eta^v (\eta(X))^c \xi^c) - \xi^c \otimes \eta^c (\varphi(X))^c$$
$$+ \xi^c \otimes \eta^c ((\eta(X))^v \xi^v) + \xi^c \otimes \eta^c ((\eta(X))^c \xi^c$$
$$= X^c - \xi^v (\eta(X))^c - (\eta(X))^v \xi^c - (\eta(X))^v \underline{\varphi^c \xi^v} - (\eta(X))^c \underline{\varphi^c \xi^c} - \xi^v (\eta(\varphi(X)))^v$$
$$+ (\eta(X))^v (\xi^v \underline{(\eta^v(\xi^v)}) + (\eta(X))^c \xi^v \underline{(\eta^v \xi^c)}) - \xi^c (\eta(\varphi(X)))^c$$
$$+ (\eta(X))^v (\xi^c \underline{(\eta^c(\xi^v)}) + (\eta(X))^c \xi^v \underline{(\eta^c(\xi^c)}).$$

If we use (1.5) and (2.1), then we get

$J^2 X^c = X^c$.

Similarly, we can prove that $J^2 X^v = X^v$, i.e. J defines an almost product (paracomplex) structure on $T(M_n)$: $J^2 = I$. On the other hand, if we now define an (1,1)-tensor field $\overline{J} = \varphi^H - \eta^v \otimes \xi^v - \eta^H \otimes \xi^H$ on $T(M_n)$, then from (1.5) and (2.2) we can show $\overline{J}^2 X^v = X^v$ and $\overline{J}^2 X^H = X^H$, which give that $\overline{J}$ is an almost product (paracomplex) structure on $T(M_n)$. Thus we have

Theorem 1. If (φ, ξ, η) is a paracontact structure on M_n, then there exists on tangent bundle $T(M_n)$ an almost product (paracomplex) structure defined by lift J ($\overline{J}$).

3. Nijenhius Tensor of Almost Paracontact Structure

Let F be an almost product structure on M_n. We say that F is integrable if the Nijenhuis tensor N_F of F is identically equal to zero. The Nijenhuis tensor N_F is defined by

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + [X, Y]$$

for any $X, Y \in \Im_0^1(M_n)$ (see, for example [1]).

Theorem 2. Let J be an almost paracomplex structure on tangent bundle $T(M_n)$ defined by $J = \varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c$. J is integrable $(N_J(X^c, Y^c) = 0)$ if $\eta(X) = \eta(Y) = 0$ and $N_\phi = 0$, where $N_\phi(X, Y) = [\phi X, \phi Y] - \phi[X, \phi Y] - \phi[\phi X, Y] + \phi^2[X, Y]$.

Proof. Calculating $N_J(X^c, Y^c)$, $J = \phi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c$, we get

$$
\begin{aligned}
N_J &= \left[JX^c, JY^c\right] - J\left[X^c, JY^c\right] - J\left[JX^c, Y^c\right] + J^2\left[X^c, Y^c\right] \\
&= \left[\left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)X^c, \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)Y^c\right] \\
&- \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[X^c, \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)Y^c\right] \\
&- \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[\left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)X^c, Y^c\right] + \left[X^c, Y^c\right] \\
&= \left[(\varphi X)^c - \xi^v(\eta X)^v - \xi^c(\eta(X))^c, (\varphi Y)^c - \xi^v(\eta Y)^v - \xi^c(\eta Y)^c\right] \\
&- \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[X^c, (\varphi Y)^c - \xi^v(\eta Y)^v - \xi^c(\eta Y)^c\right] \\
&- \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[(\varphi X)^c - \xi^v(\eta X)^v - \xi^c(\eta X)^c, Y^c\right] + \left[X^c, Y^c\right] \\
&= \left[(\varphi X)^c - \xi^v(\eta(X))^v - \xi^c(\eta(X))^c, (\varphi Y)^c\right] + \left[(\varphi X)^c - \xi^v(\eta(X))^v - \xi^c(\eta(X))^c, \xi^v(\eta Y)^v\right] \\
&- \left[(\varphi X)^c - \xi^v(\eta(X))^v - \xi^c(\eta(X))^c, \xi^c(\eta Y)^c\right] - \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[X^c, (\varphi Y)^c\right] \\
&+ \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[X^c, \xi^v(\eta Y)^v\right] + \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[X^c, \xi^c(\eta Y)^c\right] \\
&- \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[(\varphi X)^c, Y^c\right] + \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[\xi^v(\eta X)^v, Y^c\right] \\
&+ \left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)\left[\xi^c(\eta(X))^c, Y^c\right] + \left[X^c, Y^c\right] \\
&= \left[(\varphi X)^c, (\varphi Y)^c\right] - \left[\xi^v(\eta(X))^v, (\varphi Y)^c\right] - \left[\xi^c(\eta(X))^c, (\varphi Y)^c\right] - \left[(\varphi(X))^c, \xi^v(\eta Y)^v\right] \\
&+ \left[\xi^v(\eta(X))^v, \xi^v(\eta Y)^v\right] + \left[\xi^c(\eta(X))^c, \xi^v(\eta Y)^v\right] - \left[(\varphi(X))^c, \xi^c(\eta Y)^c\right] \\
&+ \left[\xi^v(\eta(X))^v, \xi^c(\eta(Y))^c\right] + \left[\xi^c(\eta(X))^c, \xi^c(\eta Y)^c\right] - \varphi^c\left[X^c, (\varphi Y)^c\right] + \xi^v o\eta^v\left[X^c(\varphi Y)^c\right] \\
&+ \xi^c o\eta^c\left[X^c(\varphi Y)^c\right] + \varphi^c\left[X^c, \xi^v(\eta Y)^v\right] - \xi^v o\eta^v\left[X^c, \xi^v(\eta Y)^v\right] - \xi^c o\eta^c\left[X^c, \xi^v(\eta Y)^v\right] \\
&+ \varphi^c\left[X^c, \xi^c(\eta Y)^c\right] - \xi^v o\eta^v\left[X^c, \xi^c(\eta Y)^c\right] - \xi^c o\eta^c\left[X^c, \xi^c(\eta Y)^c\right] - \varphi^c\left[(\varphi X)^c, Y^c\right] \\
&+ \xi^v o\eta^v\left[(\varphi X)^c, Y^c\right] + \xi^c o\eta^c\left[(\varphi X)^c, Y^c\right] + \varphi^c\left[\xi^v(\eta(X))^v, Y^c\right] - \xi^v o\eta^v\left[\xi^v(\eta X)^v, Y^c\right] \\
&- \xi^c o\eta^c\left[\xi^v(\eta X)^v, Y^c\right] + \varphi^c\left[\xi^c(\eta(X))^c, Y^c\right] - \xi^v o\eta^v\left[\xi^c(\eta(X))^c, Y^c\right] \\
&- \xi^c o\eta^c\left[\xi^c(\eta(X))^c, Y^c\right] + \left[X^c, Y^c\right] \\
&= [\varphi X, \varphi Y]^c - \left[(\eta(X))^v\xi^v, (\varphi(Y))^c\right] - \left[(\eta(X))^c\xi^c, (\varphi Y)^c\right] - \left[(\varphi(X))^c, (\eta(Y))^v\xi^v\right] \\
&+ \left[(\eta(X))^v\xi^v, \left(\eta(Y)^v\right)\xi^v\right] + \left[(\eta(X))^c\xi^c, \left(\eta(Y)^v\right)\xi^v\right] - \left[(\varphi X)^c, (\eta(Y))^c\xi^c\right] \\
&+ \left[(\eta(X))^v\xi^v, \left(\eta(Y)^c\right)\xi^c\right] + \left[(\eta(X))^c\xi^c, \left(\eta(Y)^c\right)\xi^c\right] - \varphi^c[X, \varphi Y]^c + \xi^v o\eta^v[X, \varphi Y]^c \\
&+ \xi^c o\eta^c[X, \varphi Y]^c + \varphi^c\left[X^c, (\eta(Y))^v\xi^v\right] - \xi^v o\eta^v\left[X^c, (\eta(Y))^v\xi^v\right] - \xi^c o\eta^c\left[X^c, (\eta Y)^v\xi^v\right] \\
&+ \varphi^c\left[X^c, \left(\eta(Y)^c\right)\xi^c\right] - \xi^v o\eta^v\left[X^c, (\eta(Y))^c\xi^c\right] - \xi^c o\eta^c\left[X^c(\eta(Y))^c\xi^c\right] - \varphi^c[\varphi X, Y]^c \\
&+ \xi^v o\eta^v[\varphi X, Y]^c + \xi^c o\eta^c[\varphi X, Y]^c + \varphi^c\left[(\eta(X))^v\xi^v, Y^c\right] - \xi^v o\eta^v\left[(\eta(X))^v\xi^v, Y^c\right] \\
&- \xi^c o\eta^c\left[(\eta(X))^v\xi^v, Y^v\right] + \varphi^c\left[(\eta(X))^c\xi^c, Y^c\right] - \xi^v o\eta^v\left[(\eta(X))^c\xi^c, Y^c\right] \\
&- \xi^c o\eta^c\left[(\eta(X))^c\xi^c, Y^c\right] + [X, Y]^c \\
&= [\varphi X, \varphi Y]^c - (\eta(X))^v\left[\xi^v, (\varphi(Y))^c\right] + \left((\varphi(Y))^c(\eta(X))^v\right)\xi^v \\
&- (\eta(X))^c\left[\xi^c, (\varphi(Y))^c\right] + \left((\varphi(Y))^c(\eta(X))^c\right)\xi^c \\
&- (\eta(Y))^v\left[(\varphi(X))^c, \xi^v\right] - \left[(\varphi(X))^c(\eta(Y))^v\right]\xi^v + (\eta(X))^v(\eta(Y))^v\underline{\left[\xi^v, \xi^v\right]} \\
&+ ((\eta X))^v\left(\xi^v(\eta(Y))^v\right)\xi^v - (\eta(Y))^v\left(\xi^v(\eta(X))^v\right)\xi^v
\end{aligned}
$$

$$+(\eta(X))^c(\eta(Y))^v\left[\xi^c,\xi^v\right]+(\eta(X))^c\left(\xi^c(\eta(Y))^v\right)\xi^v$$
$$-(\eta(Y))^v\left(\xi^v(\eta(X))^c\right)\xi^c-(\eta(Y))^c\left[(\varphi X)^c,\xi^c\right]-\left((\varphi X)^c(\eta(Y))^c\right)\xi^c$$
$$+(\eta(X))^v(\eta(Y))^c\left[\xi^v,\xi^c\right]+(\eta(X))^v\left(\xi^v(\eta(Y))^c\right)\xi^c$$
$$-(\eta(Y))^c\left(\xi^c(\eta(X))^v\right)\xi^v+(\eta(X))^c(\eta(Y))^c\left[\xi^c,\xi^c\right]+((\eta X))^c\left(\xi^c(\eta(Y))^c\xi^c\right)$$
$$-(\eta(Y))^c\left(\xi^c(\eta(X))^c\right)\xi^c-\varphi^c[X,\varphi Y]^c+\xi^v o\eta^v[X,\varphi Y]^c+\xi^c o\eta^c[X,\varphi Y]^c+\varphi^c(\eta(Y))^c\left[X^c,\xi^v\right]$$
$$+\varphi^c\left(X^c(\eta(Y))^v\right)\xi^v-\xi^v o\eta^v((\eta y))^v\left[X^c,\xi^v\right]-\xi^v o\eta^v\left(X^c(\eta(Y))^v\xi^v\right)-\left(\xi^c o\eta^c(\eta(Y))^v\left[X^c,\xi^v\right]\right)$$
$$-\xi^c o\eta^c\left(X^c(\eta(Y))^v\right)\xi^v+\varphi^c(\eta(Y))^c[X,\xi]^c+\varphi^c\left(X^c(\eta(Y))^c\right)\xi^c-\xi^v o\eta^v(\eta(Y))^c[X,\xi]^c$$
$$-\xi^v o\eta^v\left(X^c(\eta(Y))^c\right)\xi^c-\xi^c o\eta^c(\eta(Y))^c[X,\xi]^c-\xi^c o\eta^c\left(X^c(\eta(Y))^c\right)\xi^c-\varphi^c[\varphi X,Y]^c$$
$$+\xi^v o\eta^v[\varphi X,Y]^c+\xi^c o\eta^c[\varphi X,Y]^c+\varphi^c(\eta(X))^v\left[\xi^v,Y^v\right]-\varphi^c\left(Y^v(\eta(X))^v\right)\xi^v$$
$$-\xi^v o\eta^v(\eta(X))^v\left[\xi^v,Y^c\right]+\xi^v o\eta^v\left(Y^c(\eta(X))^v\right)\xi^v$$
$$-\xi^c o\eta^c(\eta(X))^v\left[\xi^v,Y^c\right]+\xi^c o\eta^c\left(Y^c(\eta(X))^v\right)\xi^v$$
$$+\varphi^c(\eta(X))^c\left[\xi^c,Y^c\right]-\varphi^c\left(Y^c(\eta(X))^c\right)\xi^c-\xi^v o\eta^v(\eta(X))^c[\xi,Y]^c+\xi^v o\eta^v\left(Y^c(\eta(X))^c\right)\xi^c$$
$$-\xi^c o\eta^c(\eta(X))^c[\xi,Y]^c+\xi^c o\eta^c\left(Y^c(\eta(X))^c\right)\xi^c+[X,Y]^c$$
$$=[\varphi X,\varphi Y]^c-(\eta(X))^v[\xi,\varphi Y]^v+(\varphi(Y)(\eta(x)))^v\xi^v-(\eta(X))^c[\xi,\varphi Y]^c+(\varphi(Y)(\eta(X)))^c\xi^c$$
$$-(\eta(Y))^v[\varphi(X),\xi]^v-(\varphi(X)(\eta(Y)))^v\xi^v+(\eta(X)\eta(Y))^v\underline{\left[\xi^v,\xi^v\right]}+(\eta(X))^v\underline{\left(\xi^v\left(\eta(Y)^v\right)\right)}\xi^v$$
$$-(\eta(Y))^v\underline{\left(\xi^v(\eta(X))^v\right)}\xi^v+(\eta(X))^c(\eta(Y))^c\underline{[\xi,\xi]^v}+(\eta(X))^c(\xi(\eta(Y)))^v\xi^v-(\eta(Y))^v(\xi(\eta(X)))^v\xi^c$$
$$-(\eta(Y))^c[\varphi X,\xi]^c-((\varphi X)(\eta(Y)))^c\xi^c+(\eta(X))^v(\eta(Y))^c\underline{[\xi,\xi]^v}+(\eta(X))^v(\xi(\eta(Y)))^v\xi^c$$
$$-(\eta(Y))^c(\xi(\eta(X)))^v\xi^v+(\eta(X))^c(\eta(Y))^c\underline{\left[\xi^c,\xi^c\right]}+(\eta(X))^c(\xi(\eta(Y)))^c\xi^c-(\eta(Y))^c(\xi(\eta(X)))^c\xi^c$$
$$-\varphi^c[X,\varphi Y]^c+\xi^v o\eta^v[X,\varphi Y]^c+\xi^c o\eta^c[X,\varphi Y]^c+\varphi^c(\eta(Y))^c[X,\xi]^v+\varphi^c(X(\eta(Y)))^v\xi^v$$
$$-\xi^v o\eta^v(\eta(Y))^v[X,\xi]^v-\xi^v o\eta^v(X(\eta(Y)))^v\xi^v-\xi^c o\eta^c(\eta(Y))^v[X,\xi]^v$$
$$-\xi^c o\eta^c(X(\eta(Y))^v\xi^v+\varphi^c(\eta(Y))^c[X,\xi]^c+\varphi^c(X(\eta(Y))^c\xi^c$$
$$-\xi^v o\eta^v(\eta(Y))^c[X,\xi]^c-\xi^v o\eta^v(X(\eta(Y)))^c\xi^c$$
$$-\xi^c o\eta^c(\eta(Y))^c[X,\xi]^c-\xi^c o\eta^c(X(\eta(Y)))^c\xi^c-\varphi^c[\varphi X,Y]^c$$
$$+\xi^v o\eta^v[\varphi X,Y]^c+\xi^c o\eta^c[\varphi X,Y]^c+\varphi^c(\eta(X))^v\underline{\left[\xi^v,Y^v\right]}$$
$$-\varphi^c\underline{\left(Y^v(\eta(X))^v\right)}\xi^v-\xi^v o\eta^v((\eta X))^v[\xi,Y]^v+\xi^v o\eta^v(Y(\eta(X)))^v\xi^v$$
$$-\xi^c o\eta^c(\eta(X))^v[\xi,Y]^v+\xi^c o\eta^c(Y(\eta(X)))^v\xi^v+\varphi^c(\eta(X))^c[\xi,Y]^c$$
$$-\varphi^c\left(Y(\eta(X))^c\right)\xi^c-\xi^v o\eta^v(\eta(x))^c[\xi,Y]^c+\xi^v o\eta^v(Y(\eta(X)))^c\xi^c$$
$$-\xi^c o\eta^c(\eta(X))^c[\xi,Y]^c+\xi^c o\eta^c(Y(\eta(X)))^c\xi^c+[X,Y]^c$$

From here, using (1.4), (1.5), (2.1) and $\eta(X)=0$, $\eta(Y)=0$, we have

$$N_J=[\varphi X,\varphi Y]^c-\varphi^c[X,\varphi Y]^c+\xi^v o\eta^v[X,\varphi Y]^c+\xi^c o\eta^c[X,\varphi Y]^c-\varphi^c[\varphi X,Y]^c$$
$$+\xi^v o\eta^v[\phi X,Y]^c+\xi^c o\eta^c[\phi X,Y]^c+[X,Y]^c=[\phi X,\phi Y]^c-J[X,\phi Y]^c-J[\phi X,Y]^c+[X,Y]^c$$
$$=[\varphi X,\varphi Y]^c-[\varphi[X,\varphi Y]]^c-[\varphi[\varphi X,Y]]^c+[X,Y]^c$$
$$=\left[N_\varphi[X,Y]\right]^c.$$

Thus, the proof of Theorem 3.1 is completed.

4. The Purity Conditions of g^c

Definition 1. ([1]) Let $\varphi \in \Im_1^1(M_n)$ be an affinor field on M_n. A tensor field t of type (r,s) is called pure tensor field with fespect to φ if

$$
\begin{aligned}
t(\varphi X_1, X_2,...,X_s;\ \overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}) &= t(X_1,\varphi X_2,...,X_s;\ \overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi})\\
&\ \vdots\\
&= t(X_1,X_2,...,\varphi X_s;\ \overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi})\\
&= t(X_1,X_2,...,X_s;\ {}'\varphi\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi})\\
&= t(X_1,X_2,...,X_s;\ \overset{1}{\xi},{}'\varphi\overset{2}{\xi},...,\overset{r}{\xi})\\
&\ \vdots\\
&= t(X_1,X_2,...,X_s;\ \overset{1}{\xi},\overset{2}{\xi},...,{}'\varphi\overset{r}{\xi})
\end{aligned} \tag{4.1}
$$

for any $X_1, X_2,..., X_s \in \Im_0^1(M_n)$ and $\xi^1,\xi^2,...,\xi^r \in \Im_1^0(M_n)$, where ${}'\varphi$ is the adjoint operator of φ defined by

$$({}'\varphi\xi)(X) = \xi(\varphi X) = (\xi \circ \varphi)(X).$$

In particular, from (4.1) we obtain the purity condition $g(\varphi X, Y) = g(X, \varphi Y)$ for $g \in \Im_2^0(M_n)$. The complete lift of g to the tangent bundle $T(M_n)$ is defined by $g^c(X^c, Y^c) = (g(X,Y))^c$ for any $X, Y \in \Im_0^1(M_n)$ (see [5]).

Theorem 3. Let g^c be a complete lift of associated Riemannian metric of almost paracontact structure (M_n,φ,ξ,η,g) to tangent bundle $T(M_n)$. If $\eta(X)=\eta(Y)=0$

for any $X,Y \in \Im_0^1(M_n)$, then g^c is pure with respect to the almost paracomplex structure $J = \varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c$.

Proof. Calculating the complete lift g^c, we get

$$
\begin{aligned}
& g^c(JX^c, Y^c) = g^c(X^c, JY^c)\\
& \Rightarrow g^c\left(\left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)X^c, Y^c\right) = g^c\left(X^c\left(\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c\right)Y^c\right)\\
& \Rightarrow g^c\left(\varphi^c X^c - \left(\xi^v o\eta^v\right)X^c - \left(\xi^c o\eta^c\right)X^c, Y^c\right)\\
& = g^c\left(X^c, \varphi^c Y^c - \left(\xi^v o\eta^v\right)Y^c - \left(\xi^c o\eta^c\right)Y^c\right)\\
& \Rightarrow g^c\left(\varphi^c X^c, Y^c\right) - g^c\left(\left(\xi^v o\eta^v\right)X^c, Y^c\right) - g^c\left(\left(\xi^c o\eta^c\right)X^c, Y^c\right)\\
& = g^c\left(X^c, \varphi^c Y^c\right) - g^c\left(X^c, \left(\xi^v o\eta^v\right)Y^c\right) - g^c\left(X^c, \left(\xi^c o\eta^c\right)Y^c\right)\\
& \Rightarrow g^c\left((\varphi X)^c, Y^c\right) - g^c\left(\xi^v\left(\eta^v\left(X^c\right)\right), Y^c\right) - g^c\left(\xi^c\left(\eta^c(X)^c\right), Y^c\right)\\
& = g^c\left(X^c(\varphi Y)^c\right) - g\left(X^c, \xi^v\left(\eta^v(Y)^c\right)\right) - g^c\left(X^c, \xi^c\left(\eta^c\left(Y^c\right)\right)\right)
\end{aligned}
$$

$$\Rightarrow g^c\left((\phi X)^c, Y^c\right) - g^c\left(\xi^v\left(\eta(X)\right)^v, Y^c\right) - g^c\left(\xi^c\left(\eta^c(X)^c\right), Y^c\right)$$
$$= g^c\left(X^c, (\phi Y)^c\right) - g^c\left(X^c, \xi^v\left(\eta(Y)^v\right)\right) - g^c\left(X^c, \xi^c\left(\eta(Y)\right)^c\right)$$

From here, using (1.5) and $\eta(X) = \eta(Y) = 0$, we have

$$g^c\left(\phi^c X^c, Y^c\right) = g^c\left(X^c, \phi^c Y^c\right),$$

which follows directly from of purity condition of g. On the other hand, since g is pure with respect to φ (see [2]), the proof of Theorem 4.1 is completed.

5. Tachibana Operators Applied to X^c and X^v

Definition 2. ([1]) Let $\varphi \in \Im_1^1(M_n)$, and $\Im(M_n) = \sum_{r,s=0}^{\infty} \Im_s^r(M_n)$ be a tensor alebra over R. A map $\phi_\varphi \,|_{r+s\rangle 0} : \overset{*}{\Im}(M_n) \to \Im(M_n)$ is called a Tachibana operator or ϕ_φ operatör on M_n if

(a) ϕ_φ is linear with respect to constant coefficient,

(b) $\phi_\varphi : \overset{*}{\Im}{}_s^r(M_n) \to \Im_{s+1}^r(M_n)$ for all r and s,

(c) $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \overset{C}{\otimes} L + K \overset{C}{\otimes} \phi_\varphi L$ for all $K, L \in \overset{*}{\Im}(M_n)$.

(d) $\phi_{\varphi X} Y = -(L_Y \varphi) X$ for all $X, Y \in \Im_0^1(M_n)$, where L_Y is the Lie derivation with respect to Y **,**

(e) $(\phi_{\varphi X}\eta)Y = (d(\iota_Y \eta))(\phi X) - (d(\iota_Y(\eta o \varphi)))X + \eta((L_Y\varphi)X)$
$= (\phi X(\iota_Y\eta))(\phi X) - X(\iota_{\varphi Y}\eta) + \eta((L_Y\varphi)X)$ for all $\eta \in \Im_1^0(M_n)$ and $X, Y \in \Im_0^1(M_n)$, where $\iota_Y \eta = \eta(Y) = \eta \overset{C}{\otimes} Y.$

Theorem 4. Let X^c be the complete lift of X to tangent bundle $T(M_n)$. If J is an almost paracomplex structure defined by $J = \varphi^c - \xi^v \otimes \eta^v - \xi^c \otimes \eta^c$, then the Tachibana operator associated with J and applied to X^c have an expression:

$$\phi_{JY^c} X^c = -[X, Y]^c + J[X, \phi Y]^c.$$

Proof.

$$\begin{aligned}
\phi_{JY^c} X^c &= -(L_{X^c} J)Y^c = \left(L_{X^c} JY^c - JL_{X^c} Y^c\right) \\
&= -\left([X^c, JY^c] - J[X^c, Y^c]\right) \\
&= -\left[X^c, (\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c)Y^c\right] + \left((\varphi^c - \xi^v o\eta^v - \xi^c o\eta^c)[X^c, Y^c]\right) \\
&= -\left[X^c, \varphi^c Y^c\right] + \left[X^c, (\xi^v o\eta^v)Y^c\right] + \left[X^c, (\xi^c o\eta^c)Y^c\right] + \varphi^c[X^c, Y^c] \\
&\quad - \xi^v o\eta^v[X^c, Y^c] - \xi^c o\eta^c[X^c, Y^c] \\
&= -\left[X^c, (\varphi Y)^c\right] + \left[X^c, (\xi^v o\eta^v)Y^c\right] + \left[X^c, (\xi^c o\eta^c)Y^c\right] + \varphi^c[X^c, Y^c] \\
&\quad - \xi^v o\eta^v[X, Y]^c - \xi^c o\eta^c[X, Y]^c \\
&= -[X, \varphi Y]^c + \left[X^c, (\xi^v o\eta^v)Y^c\right] + \left[X^c, (\xi^c o\eta^c)Y^c\right] + \varphi^c[X, Y]^c \\
&\quad - \xi^v o\eta^v[X, Y]^c - \xi^c o\eta^c[X, Y]^c \\
&= -[X, \varphi Y]^c + \left[X^c, (\xi^v o\eta^v)Y^c\right] + \left[X^c, (\xi^c o\eta^c)Y^c\right] + J[X, Y]^c
\end{aligned}$$

$$=-[X,\varphi Y]^c+\left[X^c,\left(\eta^v\left(Y^c\right)\right)\xi^v\right]+\left[X^c,\eta^c\left(Y^c\right)\xi^c\right]+\varphi^c[X,Y]^c$$
$$-\xi^v o\eta^v[X,Y]^c-\xi^c o\eta^c[X,Y]^c$$
$$=-[X,\varphi Y]^c+\left[X^c,(\eta(Y))^v\xi^v\right]+\left[X^c,(\eta(Y))^c\xi^c\right]+\varphi^c[X,Y]^c$$
$$-\xi^v o\eta^v[X,Y]^c-\xi^c o\eta^c[X,Y]^c$$
$$=-[X,\varphi Y]^c+(\eta(Y))^v\left[X^c,\xi^v\right]+\left(X^c(\eta(Y))^v\right)\xi^v+(\eta(Y))^c\left[X^c,\xi^c\right]+X^c(\eta(Y))^c\xi^c$$
$$+\varphi^c[X,Y]^c-\xi^v o\eta^v[X,Y]^c-\xi^c o\eta^c[X,Y]^c$$
$$=-[X,\varphi Y]^c+(\eta(Y))^v[X,\xi]^v+\left(X(\eta(Y))^v\xi^v+(\eta(Y))^c[X,\xi]^c+(X(\eta(Y)))^c\xi^c\right)$$
$$+\varphi^c[X,Y]^c-\xi^v o\eta^v[X,Y]^c-\xi^c o\eta^c[X,Y]^c$$
$$\phi_{jY^c}X^c=-[X,\varphi Y]^c+(\eta(Y))^v[X,\xi]^v+(X(\eta(Y)))^v\xi^v+(\eta(Y))^c[X,\xi]^c+(X(\eta(Y)))^c\xi^c$$
$$+\varphi^c[X,Y]^c-\xi^v o\eta^v[X,Y]^c-\xi^c o\eta^c[X,Y]^c$$
$$\phi_{(\varphi^c-\xi^v o\eta^v-\xi^c o\eta^c)Y^c}X^c=-[X,\varphi Y]^c+(\eta(Y))^v[X,\xi]^v+(X(\eta(Y)))^v\xi^v+(\eta(Y))^c[X,\xi]^c+(X(\eta(Y)))^c\xi^c$$
$$+\varphi^c[X,Y]^c-\xi^v o\eta^v[X,Y]^c-\xi^c o\eta^c[X,Y]^c$$
$$\phi_{(\varphi Y)^c-(\eta(Y))^v-(\eta(Y))^c)\xi^c}X^c=-[X,\varphi Y]^c+(\eta(Y))^v[X,\xi]^v+(X(\eta(Y)))^v\xi^v+(\eta(Y))^c[X,\xi]^c+(X(\eta(Y)))^c\xi^c$$
$$+\varphi^c[X,Y]^c-\xi^v o\eta^v[X,Y]^c-\xi^c o\eta^c[X,Y]^c$$

Putting $Y\to\varphi Y,$ by virtue of (2.1) and $(\phi^c)^2Y^c=Y^c$ we have

$$\phi_{\left((\varphi^c)^2Y^c\right)}X^c=-\left[X^c,(\varphi^c)^2Y^c\right]+\varphi^c[X,\varphi Y]^c-\xi^v o\eta^v[X,\varphi Y]^c-\xi^c o\eta^c[X,\varphi Y]^c$$
$$\phi_{JY^c}X^c=\phi_{Y^c}X^c=-[X,Y]^c+J[X,\varphi Y]^c$$

Theorem 5. Let X^v be the complete lift of X to tangent bundle $T(M_n)$. If J is an almost paracomplex structure defined by $J=\varphi^c-\xi^v\otimes\eta^v-\xi^c\otimes\eta^c$, then the Tachibana operator associated with J and applied to X^c have an expression:

$$\phi_{J(Y)^c}X^v=\phi_{Y^c}X^v=-[X,Y]^v+J[X,\varphi Y]^v.$$

Proof:

$$\phi_{J(Y)^c}X^v=-\left(L_{X^v}J\right)Y^c=-\left(L_{X^v}JY^c-JL_{X^v}Y^c\right)=-\left(\left[X^v,JY^c\right]-J\left[X^v,Y^c\right]\right)$$
$$=-\left[X^v\left(\varphi^c-\xi^v o\eta^v-\xi^c o\eta^c\right)Y^c\right]+\left(\varphi^c-\xi^v o\eta^v-\xi^c o\eta^c\right)\left[X^v,Y^c\right]$$
$$=-\left[X^v,\varphi^cY^c\right]+\left[X^v\left(\xi^v o\eta^v\right)Y^c\right]+\left[X^v\left(\xi^c o\eta^c\right)Y^c\right]+\varphi^c\left[X^v,Y^c\right]-\xi^v o\eta^v\left[X^v,Y^c\right]$$
$$-\xi^c o\eta^c\left[X^v,Y^c\right]$$
$$=-\left[X^v,(\varphi Y)^c\right]-\left[X^v,\left(\xi^v o\eta^v\right)Y^c\right]-\left[X^v,\left(\xi^c o\eta^c\right)Y^c\right]+\varphi^c[X,Y]^v+\xi^v o\eta^v[X,Y]^v$$
$$+\xi^c o\eta^c[X,Y]^v$$
$$=-[X,\varphi Y]^v+\left[X^v\left(\xi^v o\eta^v\right)Y^c\right]+\left[X^v,\left(\xi^c o\eta^c\right)Y^c\right]+\varphi^c[X,Y]^v-\xi^v o\eta^v[X,Y]^v$$
$$-\xi^c o\eta^c[X,Y]^v$$
$$=-[X,\varphi Y]^v+\left[X^v,\left(\xi^v o\eta^v\right)Y^c\right]+\left[X^v,\left(\xi^c o\eta^c\right)Y^c\right]+J[X,Y]^v$$

$$=-[X,\varphi Y]^v+[X^v,(\eta^v(Y^c))\xi^v]+[X^v(\eta^c(Y^c))\xi^c]+\varphi^c[X,Y]^v-\xi^v o\eta^v[X,Y]^v$$
$$-\xi^c o\eta^c[X,Y]^v$$
$$=-[X,\varphi Y]^v+(\eta^v(Y)^c)[X^v,\xi^v]+X^v(\eta^v(Y)^c)\xi^v+(\eta^c(Y^c))[X^v,\xi^c]$$
$$+(X^V(\eta^c(Y)^c)\xi^c)+J[X,Y]^v$$
$$=-[X,\varphi Y]^v+(\eta(Y))^v[X^v,\xi^v]+X^v(\eta(Y))^v\xi^v+(\eta(Y))^c[X,\xi]^v+(X^v(\eta(Y))^c\xi^c)$$
$$+J[X,Y]^v$$
$$\phi_{JY^c}X^v=-[X,\varphi Y]^v-(\eta(Y))^v[X^v,\xi^v]-X^v(\eta(Y))^v\xi^v-(\eta(Y))^c[X,\xi]^v$$
$$-(X(\eta(Y)))^v\xi^c+J[X,Y]^v$$
$$\phi_{(\varphi^c-\xi^v o\eta^v-\xi^c o\eta^c)Y^c}X^v=-[X,\varphi Y]^v-(\eta(Y))^v[X^v,\xi^v]-X^v(\eta(Y))^v\xi^v-(\eta(Y))^c[X,\xi]^v$$
$$-(X(\eta(Y)))^v\xi^c+J[X,Y]^v$$
$$\phi_{(\varphi Y)^c-(\eta(Y))^v-(\eta(Y))^c)\xi^c}X^v=-[X,\varphi Y]^v-(\eta(Y))^v[X^v,\xi^v]-X^v(\eta(Y))^v\xi^v-(\eta(Y))^c[X,\xi]^v$$
$$-(X(\eta(Y)))^v\xi^c+J[X,Y]^v$$

From (2.1) and $(\phi^c)^2Y^c=Y^c$, we have

$$\phi_{((\varphi^c)^2Y^c)}X^v=-[X,\phi(\phi Y)]^v+\phi^c[X,\phi Y]^v+\xi^v o\eta^v[X,\phi Y]^v+\xi^c o\eta^c[X,\phi Y]^v,$$
$$\varphi_{JY^c}X^v=\varphi_{Y^c}X^v=-[X,Y]^v+J[X,\phi Y]^v.$$

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