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On the k-Jacobsthal Numbers

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Abstract

We introduce a general Jacobsthal sequence that generalizes the classical Jacobsthal sequence. Many properties of these numbers $J_{k,n}$, $n \in N$ are deduced directly from elementary algebra in a similar way that in the case of the k–Fibonacci numbers. Finally, we will find that the Pascal triangle related with the k–Jacobsthal numbers coincides with the triangle obtained for the k–Fibonacci numbers.

Keywords: k–Fibonacci numbers, Formulas of Binnet, Catalan, D'Ocagne and convolution, Pascal triangle

1. Introduction

In this section, we introduce the k–Fibonacci numbers, defined previously by Falcon and Plaza (2007).

For any positive real number k, the k–Fibonacci sequence,say $\left\{F_{k,n}\right\}_{n\in N}$ isdefined recurrently by:

$$F_{k,n+1} = k\,F_{k,n} + F_{k,n-1} \qquad (1.1)$$

for n ≥ 1, with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

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Particular cases of the k-Fibonacci sequence are:

- If k=1, the classical Fibonacci sequence is obtained: $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. So, $\{F_n\} = \{0,1,1,2,3,5,8,...\}$
- If k=2, the classical Pell sequence appears:

$P_0 = 0, P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$. Then $P = \{0,1,2,5,12,70,169,...\}$

2. The k-Jacobsthal Numbers

In a similar form to the k-Fibonacci numbers, we define the k-Jacobsthal numbers by mean of the recurrence relation

$$J_{k,n+1} = J_{k,n} + k\,J_{k,n-1} \text{ for } n \geq 1 \tag{2.1}$$

with the initial conditions $J_{k,0} = 0$ and $J_{k,1} = 1$.

We will represent the k-Jacobsthal sequence as $J_k = \{0,1,J_{k,2},J_{k,3},...\}$

For k = 1 and k = 2, the Jacobsthal sequence J_1 coincides with the classical Fibonacci sequence and the classical Jacobsthal sequence $J = \{0,1,1,3,5,11,...\}$, respectively.

For k = 1, 2…30, all the k-Jacobsthal sequences are listed in Sloane N.J.A. from now on OEIS.In general,we will take $k \in N$ and the firstk-Jacobsthal numbers are:

Table 1

$J_{k,1} = 1$

$J_{k,2} = 1$

$J_{k,3} = 1 + k$

$J_{k,4} = 1 + 2k$

$J_{k,5} = 1 + 3k + k^2$

$J_{k,6} = 1 + 4k + 3k^2$

$J_{k,7} = 1 + 5k + 6k^2 + k^3$

2.1 The Binet Identity for the k-Jacobsthal Numbers

The solutions of the characteristic equation $r^2 = r + k$ associated to the recurrence relation (2.1) are $\sigma_1 = \frac{1+\sqrt{1+4k}}{2}$ and $\sigma_2 = \frac{1-\sqrt{1+4k}}{2}$, and consequently, the solution of the recurrence relation is $J_{k,n} = c_1\sigma_1^n + c_2\sigma_2^n$. Then:

$$n = 0 \to c_1 + c_2 = 0$$
$$n = 1 \to c_1\sigma_1 + c_2\sigma_2 = 1$$

From these equations we obtain the general term of the k-Jacobsthal sequence $J_k = \{J_{k,n}\}_{n\in N}$:

$$J_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} = \frac{1}{\sqrt{4k+1}}\left(\left(\frac{1+\sqrt{4k+1}}{2}\right)^n - \left(\frac{1-\sqrt{4k+1}}{2}\right)^n\right) \qquad (2.2)$$

If k = 1, then $\sigma_1 = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio Φ.

If $\sigma = \sigma_1$ or $\sigma = \sigma_2$ it is $\sigma^2 = \sigma + k$.

The characteristic solutions verify the following properties:

$$\sigma_1 \cdot \sigma_2 = -k \qquad \sigma_1 + \sigma_2 = 1 \qquad \sigma_1 - \sigma_2 = \sqrt{4k+1}$$
$$\sigma_1 > 1 \qquad \sigma_2 < 0 \qquad |\sigma_2| < \sigma_1$$

2.2 Two Expressions for the Positive Characteristic roots as Limits

Here, two different ways for representing the metallicmeans are given.

2.2.1 Continued Fractions

First, note that from characteristic equation $r^2 = r + k$ itis immediately obtained $\sigma_1^2 = \sigma_1 + k \to \sigma_1 = 1 + \frac{k}{\sigma_1}$, from where by repeated substitutions we have:

$$\sigma_1 = 1 + \cfrac{k}{1 + \cfrac{k}{1 + \cfrac{k}{1 + \cfrac{k}{1 + \cdots}}}}$$

Note the last continued fraction represents the positive rootof the characteristic equation, since all the terms are positive. Besides, for different values of the parameter k, we have the continued fraction corresponding of some of the most common k-Jacobsthal sequences. Then, for the classical Fibonacci sequence (k = 1), it is $\sigma_1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$

For the classical Jacobsthal sequence (k = 2), it is $\sigma_1 = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \cdots}}}}$

2.2.2 Nested Radicals

From the characteristic equation it is $r = \sqrt{1 + k\,r}$; and applying iteratively this relationwe can write $\sigma_1 = \sqrt{k + \sigma_1} = \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \cdots}}}}$ and, as in the case of continued fractions, note thisexpression corresponds to the positive characteristic root.

2.3 Ratio between two k-Jacobsthal Numbers

If *r* is a positive integer number, then $\lim_{n\to\infty}\frac{J_{k,n+r}}{J_{k,n}}=\sigma_1^r$

Proof.

$$\lim_{n\to\infty}\frac{J_{k,n+r}}{J_{k,n}}=\lim_{n\to\infty}\frac{\frac{\sigma_1^{n+r}-\sigma_2^{n+r}}{\sigma_1-\sigma_2}}{\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}}=\lim_{n\to\infty}\frac{\sigma_1^{n+r}-\sigma_2^{n+r}}{\sigma_1^n-\sigma_2^n}=\lim_{n\to\infty}\frac{\sigma_1^r-\sigma_2^r\left(\frac{\sigma_2}{\sigma_1}\right)^n}{1-\left(\frac{\sigma_2}{\sigma_1}\right)^n}=\sigma_1^r$$

because $|\sigma_2|<\sigma_1\to\lim_{n\to\infty}\left(\frac{\sigma_2}{\sigma_1}\right)^n=0$

Particularly, $\lim_{n\to\infty}\frac{J_{1,n+1}}{J_{1,n}}=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\Phi$, is the golden Ratio, $\Phi=\frac{1+\sqrt{5}}{2}$.

2.4 Relation between the Sequences $\{\sigma^n\}$ and J_k

For n ≥ 1 and if $\sigma=\sigma_1$ or $\sigma=\sigma_2$, it is $\sigma^n=J_{k,n}\sigma+J_{k,n-1}k$.

Proof. By induction. For n = 1, $\sigma=J_{k,1}\sigma+J_{k,0}k$.

For n = 2, $\sigma^2=J_{k,2}\sigma+J_{k,1}k=\sigma+k$.

Let us suppose this formula is true for *n*. Then

$$\begin{aligned}\sigma^{n+1}&=\sigma^n\sigma=\left(J_{k,n}\sigma+J_{k,n-1}k\right)\sigma=J_{k,n}\sigma^2+J_{k,n-1}k\sigma=J_{k,n}(\sigma+k)+J_{k,n-1}k\sigma\\&=(J_{k,n}+kJ_{k,n-1})\sigma+J_{k,n}k=J_{k,n+1}\sigma+J_{k,n}k\end{aligned}$$

Then, the sequence of powers $\{\sigma^n\}$ contains the k-Jacobsthal sequence as coefficients of σ and also as coefficients of the parameter k.

2.4.1 Proposition 1: Catalan Identity

For $n \geq r: J_{k,n-r}J_{k,n+r} - J_{k,n}^2 = (-k)^{n-r} J_{k,r}^2$ (2.3)

Proof. By using Equation (2.1) in the left hand side (LHS) of this relation, and taking into account $\sigma_1\sigma_2 = -k$, we obtain:

$$(LHS) = \frac{\sigma_1^{n-r} - \sigma_2^{n-r}}{\sigma_1 - \sigma_2} \cdot \frac{\sigma_1^{n+r} - \sigma_2^{n+r}}{\sigma_1 - \sigma_2} - \left(\frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}\right)^2$$

$$= \frac{\sigma_1^{2n} - \sigma_1^{n-r}\sigma_2^{n+r} - \sigma_1^{n+r}\sigma_2^{n-r} + \sigma_2^{2n} - \sigma_1^{2n} + 2\sigma_1^n\sigma_2^n - \sigma_2^{2n}}{\left(\sigma_1 - \sigma_2\right)^2}$$

$$= \frac{1}{4k+1}\left(-\left(\sigma_1\sigma_2\right)^n\left(\frac{\sigma_2}{\sigma_1}\right)^r - \left(\sigma_1\sigma_2\right)^n\left(\frac{\sigma_1}{\sigma_2}\right)^r + 2\left(\sigma_1\sigma_2\right)^n\right) = \frac{-(-k)^n}{4k+1}\left(\frac{\sigma_2^{2r} + \sigma_1^{2r}}{(\sigma_1\sigma_2)^r} - 2\right)$$

$$= -(-k)^{n-r}\frac{(\sigma_1^r - \sigma_2^r)^2}{4k+1} = -(-k)^{n-r} J_{k,r}^2$$

Note that for r = 1, Equation (2.3) gives the Simson Identity for the k-Jacobsthal sequence:

$J_{k,n-1}J_{k,n+1} - J_{k,n}^2 = (-1)^n k^{n-1}$ (2.4)

2.4.2 Proposition 2: Convolution Identity

$$J_{k,m+n} = J_{k,m+1}J_{k,n} + k\, J_{k,m}J_{k,n-1}$$

Proof. Applying the Binet formula for the k-Jacobsthal numbers to the Second Hand Right (SHR) of this relation, we have:

$$SHR = \frac{1}{(\sigma_1-\sigma_2)^2}\left[\left(\sigma_1^{m+1}-\sigma_2^{m+1}\right)\left(\sigma_1^{n}-\sigma_2^{n}\right)+k\left(\sigma_1^{m}-\sigma_2^{m}\right)\left(\sigma_1^{n-1}-\sigma_2^{n-1}\right)\right]$$

$$= \frac{1}{(\sigma_1-\sigma_2)^2}\Big[\sigma_1^{m+1+n}-\sigma_1^{m+1-n}(-k)^n-\sigma_2^{m+1-n}(-k)^n+{}_2^{m+1+n}+k\sigma_1^{m-1+n}-k\sigma_1^{m+1-n}$$

$$-k\sigma_2^{m+1-n}(-k)^{n-1}+k\sigma_2^{m-1+n}\Big]$$

$$= \frac{1}{(\sigma_1-\sigma_2)^2}\left[\sigma_1^{m-1+n}(\sigma_1^2+k)+\sigma_2^{m-1+n}(\sigma_2^2+k)\right]$$

$$= \frac{1}{(\sigma_1-\sigma_2)^2}\left[\sigma_1^{m-1+n}(\sigma_1^2-\sigma_1\sigma_2)+\sigma_2^{m-1+n}(\sigma_2^2-\sigma_1\sigma_2)\right]$$

$$= \frac{1}{(\sigma_1-\sigma_2)^2}\left[\sigma_1^{m-1+n}\sigma_1(\sigma_1-\sigma_2)+\sigma_2^{m-1+n}\sigma_2(\sigma_2-\sigma_2)\right]=\frac{1}{\sigma_1-\sigma_2}(\sigma_1^{m+n}-\sigma_2^{m+n})=J_{k,m+n}$$

Particular cases of the convolution formula:

- Even k-Jacobsthal numbers: if m = n, then $J_{k,2n}=J_{k,n+1}^2-k^2J_{k,n-1}^2$
- Odd k-Jacobsthal numbers: if m = n+1, then $J_{k,2n+1}=J_{k,n+1}^2+k\,J_{k,n}^2$
- If m = 2n, then $J_{k,3n}=J_{k,n+1}^3+k\,J_{k,n}^3-k^3J_{k,n-1}^3$

In a similar way that before the following identity is proven.

2.4.3 Proposition 3: D'Ocagne Identity

Form > n, $J_{k,m}J_{k,n+1}-J_{k,m+1}J_{k,n}=(-1)^{m-n+1}k^nJ_{k,m-n}$

2.5 Binomial formula for the k-Jacobsthal Numbers

For n ≥ 1: $J_{k,n}=\frac{1}{2^{n-1}}\sum_{j=0}^{ip}\binom{n}{2j+1}(4k+1)^j$ where *ip* is the integer part of $\frac{n-1}{2}$.

Proof. If we expand the Binnet Identity, then

$$J_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} = \frac{1}{\sqrt{4k+1}}\left[\left(\frac{1+\sqrt{4k+1}}{2}\right)^n - \left(\frac{1-\sqrt{4k+1}}{2}\right)^n\right]$$

$$= \frac{1}{\sqrt{4k+1}}\frac{1}{2^n}2\left[\binom{n}{1}\sqrt{4k+1} + \binom{n}{3}\sqrt{(4k+1)^3} + \binom{n}{5}\sqrt{(4k+1)^5} + \cdots\right] = (RHS)$$

From this formula it is easy to find any k-Jacobsthal number without having to find before the preceding terms of the k-Jacobsthal sequence.

2.6 A third Formula for the General Term of the k-Fibonacci Sequence

$$For\ \ n \geq 2: J_{k,n} = \sum_{j=0}^{ip}\binom{n-1-j}{j}k^j \qquad (2.5)$$

Proof by induction.

For n = 2it is $J_{k,2} = \sum_{j=0}^{0}\binom{1-j}{j}k^j = 1$

For n=3 it is $J_{k,3} = \sum_{j=0}^{1}\binom{2-j}{j}k^j = 1+k$

Let us suppose this formula is true until the terms $J_{k,n-1}$ and $J_{k,n}$. Now, from the definition of the k-Jacobsthal numbers, it is $J_{k,n+1} = J_{k,n} + k\,J_{k,n-1}$ so, from the induction hypothesis,

$$J_{k,n+1} = \sum_{j=0}^{ip}\binom{n-1-j}{j}k^j + k\sum_{j=0}^{ip'}\binom{n-2-j}{j}k^j = 1 + \sum_{j=0}^{ip}\binom{n-1-j}{j}k^j + \sum_{j=0}^{ip'}\binom{n-2-j}{j}k^{j+1}$$

where ip' is the integer part of $\frac{n-2}{2}$. Then, if in the last summand we replace j by j -1 then it is $J_{k,n+1} = 1 + \sum_{j=0}^{ip} \binom{n-1-j}{j} k^j + \sum_{j=0}^{ip} \binom{n-1-j}{j} k^{j+1}$.

And now, having in mind that $\binom{m}{j} + \binom{m}{j-1} = \binom{m+1}{j}$ (Graham R.L.), we obtain:

$J_{k,n+1} = 1 + \sum_{j=0}^{ip} \binom{n-j}{j} k^j + \sum_{j=0}^{ip"} \binom{n-j}{j} k^j$ where ip'' is the integer part of $\frac{n}{2}$

2.7 Sum of the first terms of the k-Jacobsthal Sequence

Binet Identity (2.2) allow us to express the sum of the firstterms of the k-Jacobsthal sequence in an easy way.

2.7.1 Proposition: Sum of first k-Jacobsthal Numbers

Let $S_{k,n}$ be the sum of the first n + 1 terms of thek-Jacobsthal sequence, that is $S_{k,n} = \sum_{j=0}^{n} J_{k,j}$. Then:

$$S_{k,n} = \frac{1}{k}\left(J_{k,n+2} - 1\right) \qquad (2.6)$$

Proof. We must take into account $\sigma_1^2 - \sigma_1 = k \to \sigma_1(\sigma_1 - 1) = k$, and, as $\sigma_1\sigma_2 = -k$, we deduce $\sigma_1 - 1 = -\sigma_2$. Similarly, $\sigma_2 - 1 = -\sigma_1$.
Then by applying the Binnet formula, it is

$$S_{k,n}=\frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}\left(\sigma_1^j-\sigma_2^j\right)=\frac{1}{\sigma_1-\sigma_2}\left(\frac{\sigma_1^{n+1}-1}{\sigma_1-1}-\frac{\sigma_2^{n+1}-1}{\sigma_2-1}\right)=\frac{1}{\sigma_1-\sigma_2}\left(\frac{\sigma_1^{n+1}-1}{-\sigma_2}+\frac{\sigma_2^{n+1}-1}{\sigma_1}\right)$$

$$=\frac{1}{\sigma_1-\sigma_2}\frac{\sigma_1^{n+2}-\sigma_1-\sigma_2^{n+2}+\sigma_2}{-\sigma_1\sigma_2}=\frac{1}{\sigma_1-\sigma_2}\frac{\sigma_1^{n+2}-\sigma_1-\sigma_2^{n+2}+\sigma_2}{k}=\frac{1}{k}\left(\frac{\sigma_1^{n+2}-\sigma_2^{n+2}}{\sigma_1-\sigma_2}-1\right)$$

$$=\frac{1}{k}\left(J_{k,n+2}-1\right)$$

As particular cases, for k = 1, the sum of the first classical Fibonacci numbers is $S_{1,n}=F_{n+2}-1$, and for k = 2, for the classical Jacobsthal numbers it is $S_{2,n}=\frac{1}{2}\left(J_{n+2}-1\right)$

2.7.2 The Pascal 2 – Triangle

From Table 1 we can see the coefficients of the powers of *k* in the k-Jacobsthal numbers are the same that in the expressions of the k-Fibonacci numbers, and, consequently, these coefficients form the same Pascal 2--triangle (Falcon S. and Plaza A. (2))

3. Generating Functions for the k-Jacobsthal Sequences

In this section, the generating functions for the k-Jacobsthal sequences are given. As a result, k-Jacobsthal sequences are the coefficients of the corresponding generating function.

Let us suppose the k-Jacobsthal numbers are thecoefficients of a potential series centred at the origin, and letus consider the corresponding analytic function $j_k(x)$. Thefunction defined in such a way is called the generating functionof the k-Jacobsthal numbers.

So, $j_k(x)=J_{k,0}+J_{k,1}x+J_{k,2}x^2+J_{k,3}x^3+\cdots+J_{k,n}x^n+\cdots$

And then,

$$x\, j_k(x) = J_{k,0}x + J_{k,1}x^2 + J_{k,2}x^3 + J_{k,3}x^4 + \cdots + J_{k,n-1}x^n + \cdots$$
$$x^2 j_k(x) = J_{k,0}x^2 + J_{k,1}x^3 + J_{k,2}x^4 + J_{k,3}x^5 + \cdots + J_{k,n-2}x^n + \cdots$$

From where, since $J_{k,j} = J_{k,j-1} + k\, J_{k,j-2}$ with $J_{k,0} = 0,\ J_{k,1} = 1$, we obtain
$\left(1 - x - x^2\right) j_k(x) = J_{k,1}x + (J_{k,2} - J_{k,1})x^2 = x$

So the generating function for the k-Jacobsthal sequence $J_k = \left\{J_{k,n}\right\}_{n\geq 0}$ is
$j_k(x) = \dfrac{x}{1 - x - k\,x^2}$.

Note that by doing the quotient of the generating function a powerseries, centered at the origin appears,

$j_k(x) = x + x^2 + (1+k)x^3 + (1+2k)x^4 + (1+3k+k^2)k^5 + \cdots$ where the coefficients of the powers of *k* areprecisely those in the Pascal 2 – triangle.

4. Conclusions

New generalized k-Jacobsthal sequences have been introduced andstudied. Several properties of these numbers are deduced and relatedwith the so-called Pascal 2-triangle. In addition, thegenerating functions for these k-Jacobsthal sequences have been given.

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