Generating Functions for $P(n, p, *)$ and $P(n, *, p)$

Sabuj Das[1]

Haradhan Kumar Mohajan[2]

Abstract

This paper shows how to prove the Theorem $P(n, p, *) = P(n, *, p)$, i.e., the number of partitions of n into p-parts is equal to the number of partitions of n having largest part p.

Key Words: Irrelevant, decreasing order, *p*-parts

1. Introduction

We give some definitions of a partition, $P(n, p, \le q)$, $P(n, \le q, p)$, $P(n, p, *)$ and $P(n, *, p)$. We generate the generating functions for $P(n, p, \le q)$, $P(n, \le q, p)$, $P(n, p, *)$ and $P(n, *, p)$ and prove the theorem $P(n, p, *) = P(n, *, p)$ by graphically. Finally we give a numerical example when n = 8.

2. Definitions

Partition: A partition of a number is a representation of n as the sum of any number of positive integral parts. Thus, 5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1. The order of the parts is irrelevant, so that parts to be arranged in decreasing order of magnitude, we denote by $P(n)$, the number of partitions of n. Thus, $P(5) = 7$.

$P(n, p, \le q)$: The number of partitions of n into p-parts, none of which exceeds q.

$P(n, \le q, p)$: The number of partitions of n into p or any smaller number of parts, the greatest of which is p.

$P(n, p, *)$**:** The number of partitions of n into p-parts.

$P(n, *, p)$: The number of partitions of n having largest part p.

3. Generating Functions for $P(n, p, \le q)$

The Generating functions for $P(n, p, \le q)$ is of the form [1]:

$$\frac{1}{(1 - zx)(1 - zx^2)\ldots(1 - zx^q)}$$

$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, p, \le q) \right\}. \qquad (1)$$

[1] Senior Lecturer, Department of Mathematics. Raozan University College, Bangladesh.
[2] Premier University, Chittagong, Bangladesh.

It is convenient to define $P(n, p, \leq q) = 0$ if $n < p$. The coefficient $P(n, p, \leq q)$ is the number of partitions of *n* into *p*-parts, none of which exceeds *q*.

Again the generating function for $P(n, \leq q, p)$ is of the form;

$$\frac{1}{(1-zx)(1-zx^2)\ldots(1-zx^q)}$$
$$= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, \leq q, p) \right\}. \qquad (2)$$

The proof of the Theorem $P(n, p, \leq q) = P(n, \leq q, p)$ is given in Hardy and Wright [2]. If $q \to \infty$, in (1), such as $\lim_{q\to\infty} x^q = 0$ when $|x| < 0$, then (1) becomes;

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)\ldots\infty}$$
$$= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, p, *) \right\} \qquad (3)$$

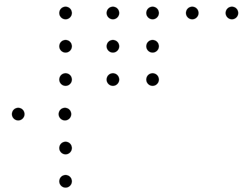
where the coefficient $P(n, p, *)$ is the number of partitions of *n* into *p*-parts. Again (2) becomes;

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)\ldots\infty}$$
$$= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, *, p) \right\} \qquad (4)$$

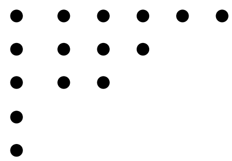
where the coefficient $P(n, *, p)$ is the number of partitions of *n* having largest part *p*.

Now we can consider a Theorem as follows:

Theorem: $P(n, p, *) = P(n, *, p)$ i.e., the number of partitions of *n* into *p*-parts is equal to the number of partitions of *n* having largest part *p*.

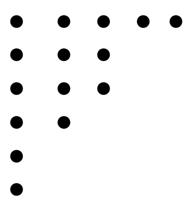
Proof: We establish a one-to-one correspondence between the partitions enumerated by $P(n, p, *)$ and those enumerated by $P(n, *, p)$. Let $n = a_1 + a_2 + \ldots + a_p$ be a partition of *n* into *p*-parts. We transfer this into a partition of *n* having largest part *p* and can represent a partition of 15 graphically by an array of dots or nodes such as, *A* 

The dots in a column correspond to a part. Thus *A* represents the partition 6+4+3+1+1 of 15. We can also represent *A* by transposing rows and columns in which case it would represent the partition graphically as conjugate of *A*.

B = Conjugate of *A* 

The dots in a column correspond to a part, so that it represents the partition 5+3+3+2+1+1 of 15. Such pair of partitions are said to be conjugate. The number of parts at 1st one portion is equal to the largest part of 2nd one partition, so that our corresponding is one-to-one.

Conversely, we can represent the partition *B* = conjugate of *A*, by transposing rows and columns, in which case it would represent the same partition like *A*, so we can say that the largest part of the partition is equal to the number of parts of the partition, then our corresponding is onto, i.e., the number of partitions of *n* into *p*-parts is equal to the number of partitions of *n* having largest part *p*. Consequently,



$P(n, p, *) = P(n, *, p)$.

Hence the Theorem.

4. A Numerical Example When n = 8

The list of partitions of 8 into 4 parts is given as follows:

5+1+1+1 = 4+2+1+1 = 3+3+1+1 = 3+2+2+1= 2+2+2+2. The number of such partitions is 5 i.e., $P(8,4,*) = 5$.

Again the list of partitions of 8 having largest part 4 is given by;

4+4 = 4+3+1 = 4+2+1+1 = 4+1+1+1+1 = 4+2+2.

So the number of such partitions is 5, i.e., $P(8,*,4) = 5$. Here 4+4, 4+3+1, 4+2+1+1, 4+1+1+1+1 and 4+2+2 are the conjugate partitions of 2+2+2+2+, 3+2+2+1, 4+2+1+1, 5+1+1+1 and 3+3+1+1 respectively. Thus the number of partitions of 8 into parts, the largest of which is 4 i.e., $P(8,4,*) = P(8,*,4)$.

5. Conclusion

For any positive integer of n, we can verify the Theorem $P(n, p,*) = P(n,*, p)$. We have already satisfied the Theorem when $n = 8$.

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